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Symmetric positive definite (SPD)

$A = A^T \quad x^T A x > 0 \quad \forall x \in \mathbb{R}^n \quad A = V \Lambda V^T \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \lambda_i > 0$

Cholesky factorization: $A = LL^T$ (no pivoting)

$$A = \begin{pmatrix} a_{11} & a_{21}^T \\ a_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21}^T \\ 0 & L_{22}^T \end{pmatrix}$$

$a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}}$

$a_{21} = l_{21} l_{11} \Rightarrow l_{21} = a_{21}/\sqrt{a_{11}}$

$A_{22} = L_{22} L_{22}^T + l_{21} l_{21}^T$

$$\underbrace{A_{22} - l_{21} l_{21}^T}_{S} = L_{22} L_{22}^T$$

✓ Claim: $a_{11} > 0$

Proof: $e_1^T A e_1 = a_{11} > 0$

Claim: S is SPD

① A SPD $\Leftrightarrow$ A^{-1} SPD

② M SPD $\Leftrightarrow$ any block SPD

$$\begin{pmatrix} x_B^T & 0 \end{pmatrix} \begin{pmatrix} M_{BB} & m \\ m & m \end{pmatrix} \begin{pmatrix} x_B \\ 0 \end{pmatrix} = x_B^T M_{BB} x_B > 0$$

$$(Px)^T = (x^T P^T) M (Px)$$

③ $S^{-1} = (A^{-1})_{(2,2)}$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$A_{11} B_{12} + A_{12} B_{22} = 0$$

$$A_{21} B_{12} + A_{22} B_{22} = I$$

$$-A_{21} A_{11}^{-1} A_{12} B_{22} + A_{22} B_{22} = I$$

$$(A_{22} - A_{21} A_{11}^{-1} A_{12}) B_{22} = I$$

$$A \text{ SPD} \Rightarrow A^{-1} \text{ SPD}$$

$$\Rightarrow B_{22} = S^{-1} \text{ SPD}$$

$$\Rightarrow S \text{ SPD}$$

Do we need pivoting?

$$(A + \delta A)\hat{x} = b \qquad \frac{\|\delta A\|}{\|A\|} = O(\varepsilon)$$

Cholesky

Some intuition

① $A = LL^T \quad L = U \Sigma V^T$

$$A = U \Sigma V^T V \Sigma U^T = U \Sigma^2 U^T$$

$$\|A\|_2 = \sigma_1^2 \quad \|L\|_2 = \sigma_1$$

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