HW for 2019-09-09

(due: 2019-09-16)

You may (and should) talk about problems with each other and with me, providing attribution for any good ideas you might get. Your final write-up should be your own.

1: Norm! Show the following for $A = xy^T$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$:

- $\|A\|_1 = \|x\|_1 \|y\|_\infty$
- $\|A\|_\infty = \|x\|_\infty \|y\|_1$
- $\|A\|_F = \|x\|_2 \|y\|_2$
- $\|A\|_2 = \|x\|_2 \|y\|_2$

2: Frobenius fun The *Frobenius inner product* over $\mathbb{R}^{m \times n}$ is

$$\langle X, Y \rangle_F = \sum_{i=1}^m \sum_{j=1}^n x_{ij} y_{ij} = \operatorname{tr}(Y^T X)$$

The associated norm (the Frobenius norm) is a consistent matrix norm, but not an operator norm.

- Argue that the Frobenius norm cannot be an operator norm. *Hint:* What is the Frobenius norm of the identity?
- Show that if H is symmetric ($H = H^T$) and S is skew $S = -S^T$, then $\langle H, S \rangle_F = 0$. Argue that therefore $\|H + S\|_F^2 = \|H\|_F^2 + \|S\|_F^2$.
- Using the cyclic property of traces ($\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$), show that
  $$\langle AX, BY \rangle_F = \langle B^T A, Y X^T \rangle_F,$$
  assuming the dimensions of the matrices make sense.

3: Goodness gradients Write the directional derivative of $\|Ax\|^2$ as

$$\delta \left[\|Ax\|^2 \right] = (\delta x)^T g + \langle \delta A, G \rangle_F.$$