CS 621: Final Exam

Monday, December 12, 2005

9:00-11:30

Most problems ask for a solution in the form of a MATLAB function. Points will not be deducted for minor syntax errors. It is important to explain why your solution works.

Problem 1	15 points	
Problem 2	20 points	
Problem 3	20 points	
Problem 4	15 points	
Problem 5	15 points	
Problem 6	15 points	
	100 points	

NAME : ____

1. (a) (7 points) Complete the following function so that it performs as specified:

```
  function [W Y] = Update(B,C,u,v)
% B and C are n-by-k matrices
% u and v are n-by-1 vectors
% W and Y are n-by-(k+1) matrices so that (I + W*Y') = (I + B*C')*(I + u*v')
```

How many flops does your implementation require assuming that $n >> k$?

(b) (8 points) Suppose $A \in \mathbb{R}^{m \times n}$ has rank n, $b \in \mathbb{R}^m$, and $\lambda \in \mathbb{R}$. Show how to use the SVD of A to compute $\| x(\lambda) \|_2^2$ where $x(\lambda) \in \mathbb{R}^n$ solves the problem

$$\min_{x \in \mathbb{R}^n} \left\| \begin{bmatrix} A \\ \lambda I_n \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2$$

2. (20 points) Recall that if $C \in \mathbb{R}^{n \times n}$ then `[L,U,P] = lu(C)` returns the factorization $PC = LU$ where $L \in \mathbb{R}^{n \times n}$ is unit lower triangular, $U \in \mathbb{R}^{n \times n}$ is upper triangular and $P \in \mathbb{R}^{n \times n}$ is a permutation matrix.

It is known that μ is a non-repeated eigenvalue of $A \in \mathbb{R}^{n \times n}$ and that there is a unique $x \in \mathbb{R}^n$ so that

- $Ax = \mu x$
- $x_i > 0, i = 1{:}n$
- $\| x \|_2 = 1$

and a unique $y \in \mathbb{R}^n$ so that

- $A^T y = \mu y$
- $y_i > 0, i = 1{:}n$
- $\| y \|_2 = 1.$

Write a MATLAB function `[x,y] = TwoVecs(A,mu)` that computes x and y. Make effective use of `lu`. Ignore round-off and be efficient.

3. (a) (10 points) Assume that

$$A = \begin{bmatrix} d_1 & f^T \\ f & \mathrm{diag}(d_2, \ldots, d_n) \end{bmatrix} \qquad f \in \mathbb{R}^{(n-1)}$$

is positive definite. Show how to determine

$$G = \begin{bmatrix} \delta_1 & 0 \\ g & \Delta \end{bmatrix} \qquad g \in \mathbb{R}^{(n-1)}$$

with $\Delta = \mathrm{diag}(\delta_2, \ldots, \delta_n)$ so that $A = GG^T$.

(b) (10 points) Given a unit 2-norm vector $v \in \mathbb{R}^{(n-1)}$, show how the SVD can be used to determine c and s with $c^2 + s^2 = 1$ so that the solution to

$$\begin{bmatrix} \delta_1 & 0 \\ g & \Delta \end{bmatrix} \begin{bmatrix} \mu \\ z \end{bmatrix} = \begin{bmatrix} c \\ sv \end{bmatrix}$$

has maximum 2-norm.

4. (15 points) In each of the following $\mathbf{u}$ designates the unit roundoff. Brief "order-of-magnitude" answers are expected.

(a) Assume that A is symmetric and positive definite. How in the worst case might the solution to $Ax = b$ change if the entries in A are perturbed by $O(\mathbf{u}\| A \|)$?

(b) How in the worst case might the solution to the full rank least squares problem $\min\| Ax - b \|_2$ change if the entries in A are perturbed by $O(\mathbf{u}\| A \|)$?

(c) Assume that λ is a nonrepeated eigenvalue of a symmetric matrix A. How in the worst case might λ change if the entries in A are perturbed by $O(\mathbf{u}\| A \|)$?

(d) Assume that λ is a nonrepeated eigenvalue of an unsymmetric matrix A. How in the worst casemight λ change if the entries in A are perturbed by $O(\mathbf{u}\| A \|)$?

(e) Assume that A has a unique minimum singular value. How in the worst case might the corresponding right singular vector change if the entries in A are perturbed by $O(\mathbf{u}\| A \|)$?

5. (15 points) Complete the following function so that it performs as specified.

```
  function alpha = ConstrainedMax(A,u)
% A is an n-by-n symmetric matrix
% u is a unit 2-norm n-vector
% Let R be the Rayleigh quotient R(x) = (x'*A*x)/(x'*x)
% alpha is the maximum value of R(x) subject to the constraint that x is orthogonal to u.
```

You may use `QR` and `schur`. Do not worry about efficiency in your implementation. But write a few sentences indicating how you would implement `ConstrainedMax` using the Lanczos method if A was large and sparse and efficiency *was* important.

6. (a)(5 points) Suppose $Q^T AQ = T$ is the Schur decomposition of $A \in \mathbb{R}^{n\times n}$ and that all of A's eigenvalues are real. Explain how the columns of Q define a nested sequence of invariant subspaces.

(b)(10 points) Complete the following function so that it performs as specified

```
  function [c,s] = ReOrder(T)
% T is a 2-by-2 upper triangular matrix.
% c^2 + s^2 = 1 with the property that if
%           S = [c  s; -s  c]'*T*[c  s; -s  c]
%
% then S is upper triangular with S(1,1) = T(2,2) and S(2,2) = T(1,1)
```

Your implementation should not make use of `schur`, `eig`, `qr` or `svd`. It must handle all possible T in a numerically reliable fashion.