CS 621: Assignment 2

Due: Wednesday, September 26, 2007 (In Lecture or 5153 Upson by 4pm)

Scoring for each problem is on a 0-to-3 scale (3 = complete success, 2 = overlooked a small detail, 1 = germ of the right idea, 0 = missed the point of the problem.) One point will be deducted for insufficiently commented code. Unless otherwise stated, you are expected to utilize fully MATLAB'S vectorizing capability subject to the constraint of being flop-efficient. Test drivers and related material are posted on the course website `http://www.cs.cornell.edu/courses/cs621/2007fa/`. For each problem requiring a MATLAB function, submit output and a listing of all scripts/functions that you had to write in order to produce the output. You are allowed to discuss *background* issues with other students, but the codes you submit must be your own.

P1. (A Constrained Condition Number)

Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular. It can be shown that

$$\kappa_2(A) \;=\; \max_{\|E\|_2 \le \|A\|_2} \lim_{t\to 0} \frac{\|(A+tE)^{-1} - A^{-1}\|_2}{t\|A^{-1}\|_2}$$

The recipe on the right is a normalized Frechet derivative. This shows how the condition number measures rate-of-change in the inverse.

Now suppose

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad A_{11} \in \mathbb{R}^{p\times p},\; A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$$

and let $\mathbf{A}$ be the set of n-by-n matrices that have this block structure *with only the (1,2) block being nonzero.* Show that the constrained condition number

$$\kappa_2(A,\mathbf{A}) \;=\; \max_{\substack{\|E\|_2 \le \|A_{12}\|_2 \\ E\in\mathbf{A}}} \lim_{\mathbf{t}\to \mathbf{0}} \frac{\|(A+tE)^{-1} - A^{-1}\|_{\mathbf{2}}}{\mathbf{t}\|A^{-1}\|_{\mathbf{2}}}$$

satisfies

$$\frac{\kappa_2(A,\mathbf{A})}{\kappa_2(A)} \;=\; \frac{\sigma_{min}(A)}{\sigma_{min}(A_{11})}\frac{\sigma_{min}(A)}{\sigma_{min}(A_{22})}\frac{\sigma_{max}(A_{12})}{\sigma_{max}(A)}$$

Here, $\sigma_{min}(\cdot)$ and $\sigma_{max}(\cdot)$ designate the smallest and largest singular value. Make free use of any Chapter 2 SVD facts.

P2. (Many 2-by-2 Problems)

Complete the following function so that it performs as specified

```
  function X = MultiSolve(A,B)
% A is an m-by-4 matrix with the property that the matrices
%       [ A(i,1)  A(i,2); A(i,3) A(i,4)] i=1:m
% are nonsingular.
% B is an m-by-2 matrix.
% X is an m-by-2 matrix with the property that
%      [ A(i,1)  A(i,2); A(i,3) A(i,4)] [X(i,1);X(i,2)]  = [B(i,1);B(i,2)]
% i=1:m
```

Your implementation should be flop efficient, involve no loops, and should be numerically equivalent to

```
for i=1:m
   [L,U,P] = lu([A(i,1) A(i,2); A(i,3) A(i,4)]);
   z = U\(L\(P*[B(i,1); B(i,2)]));
   X(i,:) = z';
end
```

You will want to use theMATLAB `find` function to implement the vectorized pivot calculation. A test script `A2P2` is on the website.

P3. (An Inverse Markov Problem)

We say that $v \in \mathbb{R}^n$ is a probability vector if $v_i \geq 0$ and $v_1 + \cdots + v_n = 1$. Suppose $A \in \mathbb{R}^{n \times n}$ is stochastic, i.e., its columns are probability vectors. Assume b, c, and d are given probability vectors. Note that for a given k that satisfies $1 \leq k \leq n$, the matrix

$$A(\lambda) = A + \lambda\,(c - A(:,k))\, e_k^T \qquad e_k = I_n(:,k)$$

is also stochastic provided $0 \leq \lambda \leq 1$. (All we are doing here is modifying the k-th column of A.) Subject to the constraint $0 \leq \lambda \leq 1$ how would you choose λ to minimize $\| x - d \|_2$ where x solves $A(\lambda)x = b$? Use the Sherman-Morrison formula. Write a MATLAB function `lambda = OptInvMarkov(A,b,c,d,k)`. You may assume that A is nonsingular. A test script `A2P3` is on the website.

GTD2. (Approximate Exponential of an Intensity Matrix)

In this problem, the notation $M > 0$ where M is a matrix means that $m_{ij} > 0$ for all i and j. Likewise for $M < 0$, $M \geq 0$ and $M \leq 0$.

If $A \in \mathbb{R}^{n \times n}$ then define the matrix exponential of A by

$$e^{At} \;=\; \sum_{k=0}^{\infty} \frac{(At)^k}{k}$$

Recall that $x(t) = e^{At}x_0$ solves the initial value problem $\dot{x} = Ax,\ x(0) = x_0$.

We say that $A \in \mathbb{R}^{n \times n}$ is an *intensity matrix* if its column sums are zero, its off-diagonal entries are positive, and its diagonal entries are negative. This means that for $j = 1{:}n$

$$-a_{jj} = \sum_{i \neq j}^{n} a_{ij} > 0.$$

Denote the diagonal and off-diagonal parts of an intensity matrix by D and E respectively.

If A is an intensity matrix, then e^{At} is stochastic for all $t > 0$. That is, it has unit column sums and its entries are nonnegative. To see this, let e be the vector of all ones. Clearly $e^T e^{At} = e^T$ thereby confirming that the exponential has unit column sums. To show that its entries are non-negative, look at

$$e^{At} \;=\; I + (D+E)t + O(t^2) \;=\; (I + Dt) + Et + O(t^2)$$

There clearly exists a small t_* so that $e^{At} > 0$ for all $0 < t \leq t_*$. Thus, if $t/k < t_*$, then it follows that $e^{At} = (e^{At/k})^k > 0$. The exponentiation of intensity matrices comes up in the study of continuous Markov processes.

The (q,q) Pade approximation to the matrix exponential is given by

$$R_{qq}(A) = \left(\sum_{k=0}^{q} c_k (-A)^k \right)^{-1} \left(\sum_{k=0}^{q} c_k A^k \right) \qquad c_k = \frac{(2q-k)!q!}{(2q)!k!(q-k)!}$$

Develop a conjecture of the form "If the intensity matrix A statisfies blah blah blah, then $R_{qq}(A)$ is stocahstic." Back up your conjecture with experimental results and (of course) any theorems that you can prove (even for the $n = 2$ case). You may assume that $q \leq 8$ and $n \leq 1000$.