CS 621: Midterm Quiz SOLUTIONS

Approximate letter grade intervals. A = [90,100], A- = [80-89], B = [70-79], C = [60-69]

1. Complete the following function so that it performs as specified:

```
  function [x,y] = HessSolver(H,b)
% H is an n-by-n nonsingular matrix with H(i,j) = 0 whenever i>j+1.
% b is a column n-vector.
% x solves Hx = b and y solves H'*y = b.
```

Make use of the MATLAB `lu` function `[L,U,P] = lu(·)`. Use \ to solve triangular systems.

Note that $PH = LU$ is $O(n^3)$ while $PH^T = LU$ is $O(n^2$. No one observed this so -7 for everybody.

```
%7 points
[L,U,P] = lu(H');

%9 points
y = U\(L\(P*b));     %Lw = Pb, Uy = w

%9 points
x = P*(L'\(U'\b));   % HP' = U'L' so  U'z = b, L'w = z, x = P'*w
```

2. Complete the following MATLAB function so that it performs as specified:

```
  function [x,y] = Solver(A,b,c)
% A is n-by-n, symmetric, and positive definite.
% b is n-by-1 and c is (n-1)-by-1
% x satisfies Ax = b and y satisfies A(1:n-1,1:n-1)y = c
```

Use \ to solve triangular systems and assume the availability of a function `G = chol(A)` that returns the Cholesky factor of a symmetric positive definite matrix.

Since

$$A = \begin{bmatrix} A_1 & v \\ v^T & \alpha \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ w^T & \mu \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ w^T & \mu \end{bmatrix}^T = GG^T$$

it follows that $G(1{:}n-1, 1{:}n-1)$ is the Cholesky factor of $A(1{:}n-1, 1; n-1)$. (13 points) Thus

```
[n,n] = size(A);
G = chol(A);
% 12 points
x = G'\(G\b);
% 13 points
y = G(1:n-1,1;n-1)'\(G(1:n-1,1:n-1)\c);
```

There is also a fullcredit solution that does `G = chol(A(1:n-1,1:n-1))` and then gets the "big G" by comparing blocks $A = G_{big}G_{big}^T$

3. Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular that $A = U\Sigma V^T$ is its SVD.

Facts:

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T \qquad\qquad A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T = \sum_{i=1}^{n} \sigma_i v_i u_i^T/\sigma_i$$

and so

$$x = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i$$

(a) Give an example of a vector $b \in \mathbb{R}^n$ such that if $Ax = b$ then $\| x \|_2 = \| b \|_2/\sigma_{min}$ where σ_{min} is the smallest singular value of A.

If $b = u_n$ then $x = v_n/\sigma_n$ has the required property. (12 points)

(b) Give an example of a matrix $E \in \mathbb{R}^{n\times n}$ so that $\kappa_2(A + E) = 2\kappa_2(A)$. Here, $\kappa(\cdot)$ is the 2-norm condition. (13 points)

let

$$E = -(\sigma_n/2)u_n v_n^T$$

4. Throughout this problem you may use \ to solve triangular linear systems. Assume that $S, T \in \mathbb{R}^{n\times n}$ are upper triangular and $B \in \mathbb{R}^{n\times n}$. We want to solve the linear system $SX - XT = B$ for $X \in \mathbb{R}^{n\times n}$. Assume that the system is nonsingular.

(a) How would you compute $X(:,1)$? (5 points)

Compare first columns: $SX(:,1) - T(1,1)X(:,1) = B(:,1)$ and so

```
X(:,1) = (S - T(1,1)*eye(n,n))\B(:,1);
```

(b) How would you compute $X(:,2)$ given that you have computed $X(:,1)$? (5 points)

Compare second columns: $SX(:,2) - T(1,2)X(:,1) - T(2,2)X(:,2) = B(:,2)$ and so

```
X(:,2) = (S - T(2,2)*eye(n,n))\(B(:,2)+T(1,2)*X(:,1));
```

(c) Complete the following MATLAB function so that it performs as specified: (15 points)

```
  function X = Sylvester(S,T,B)
% S and T are n-by-n upper triangular matrices.
% B is n-by-n
% X is n-by-n and satisfies SX - XT = B


  [n,n] = size(S)
  X = zeros(n,1);;
  for j=1:n
     d = B(:,j);
     for i=1:j-1
        d = d + T(i,j)*X(:,i);
     end
     X(:,j) = (S - T(j,j)*eye(n,n))\d;
  end
```