CS 621: Assignment 5

Due: Friday, November 18, 2005 (In Lecture or 4130 Upson by 4pm)

Scoring for each problem is on a 0-to-3 scale (3 = complete success, 2 = overlooked a small detail, 1 = germ of the right idea, 0 = missed the point of the problem.) One point will be deducted for insufficiently commented code. Unless otherwise stated, you are expected to utilize fully MATLAB'S vectorizing capability subject to the constraint of being flop-efficient. Test drivers and related material are posted on the course website `http://www.cs.cornell.edu/courses/cs621/2005fa/`. For each problem submit output and a listing of all scripts/functions that you had to write in order to produce the output. You are allowed to discuss *background* issues with other students, but the codes you submit must be your own.

P1. (Nearest Symmetric Rank-1)

If $A \in \mathbb{R}^{n\times n}$ has SVD $A = U\Sigma V^T$, then the problem

$$\min_{\mathrm{rank}(S)=1} \quad \| A - S \|_F$$

is solved by $S = \sigma_1 u_1 v_1$ where $u_1 = U(:,1)$, $v_1 = V(:,1)$, and $\sigma_1 = \Sigma(1,1)$. Figure out how to solve

$$\min_{\substack{\mathrm{rank}(S)=1 \\ S = S^T}} \quad \| A - S \|_F$$

Note that if $S \in \mathbb{R}^{n\times n}$ is a symmetric rank-1 matrix then either $S = vv^T$ or $S = -vv^T$ for some $v \in \mathbb{R}^n$. Complete the following function so that it performs as specified:

```
  function [alpha,u] = NearestSym(A)
% A is n-by-n
% u is an n-by-1 unit 2-norm vector and alpha is a scalar such that S = alpha*u*u'
% minimizes norm(A-S,'fro') over all symmetric rank-1 matrices.
```

If you need eigenvalues and eigenvectors of a symmetric matrix, use the MATLAB function `schur`. Test your implementation with the script `P1`. *Hint.* To derive your method you'll have to use some properties of the trace:

- If $C \in \mathbb{R}^{n\times n}$ then $\mathrm{tr}(C) = \sum_{i=1}^n c_{ii} \;=\; \sum_{i=1}^n \lambda_i(C)$
- If $C \in \mathbb{R}^{m\times n}$ then $\| C \|_F^2 = \mathrm{tr}(C^TC)$.
- If $B, C \in \mathbb{R}^{n\times r}$ then $\mathrm{tr}(B^TC) = \mathrm{tr}(CB^T)$.
- If $C, X \in \mathbb{R}^{n\times n}$ and X is nonsingular, then $\mathrm{tr}(X^{-1}CX) = \mathrm{tr}(C)$.

I want to see your derivation, either via comments in your code or in written form.

P2.(Left Half Plane Eigenvalues of an Orthogonal Matrix)

If $U \in \mathbb{R}^{n\times n}$ is orthogonal, then its eigenvalues are on the unit circle. We wish to compute the number of its eigenvalues that are in the open left half plane. Note that

$$Ux = \lambda x \qquad \Leftrightarrow \qquad U^Tx = \frac{1}{\lambda}x$$

This implies that if $\lambda \in \lambda(U)$ then $\mathrm{Re}(\lambda) \in \lambda((U+U^T)/2)$. Complete the following function so that it performs as specified

```
  function m = nLeftHalfPlaneEigs(U)
% U is a real n-by-n orthogonal matrix
% m is the number of U's eigenvalues that are located in the open left half plane.
```

You are allowed to use the MATLAB function `hess` but not the function `schur`. Test your implementation with the script `P2`. Use sensible tolerances for detecting numerical zeros, e.g., $|a_{ij}| \leq \texttt{eps}\| A \|_1$

GTD5. (Constant Diagonal)

If

$$A = \begin{bmatrix} w & x \\ x & z \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$, then

$$B(\theta) \;=\; Q^T AQ \;=\; \begin{bmatrix} c^2w - 2csx + s^2z & sc(w-z) + (c^2 - s^2)x \\ sc(w-z) + (c^2 - s^2)x & s^2w + 2csx + c^2z \end{bmatrix}$$

It can be shown that

$$\lambda_2(A) \leq b_{11}(\theta) \leq \lambda_1(A)$$

and that the upper and lower bounds are achievable.(Why?) When that happens $B = Q^TAQ$ is diagonal and we have a Schur decomposition. (Why?)

Note that if $\tau \in \mathbb{R}$ satisfies $\lambda_2(A) \leq \tau \leq \lambda_1(A)$, then there is a θ so that $b_{11}(\theta) = \tau$. In other words, we can put a specified value in the (1,1) entry of Q^TAQ as long as that value is in the interval $[\lambda_2(A), \lambda_1(A)]$. (Why?) If τ is the midpoint of this interval, then $b_{11}(\theta) = b_{22}(\theta)$. (Why?)

Stepping up to the n-by-n case, if $A \in \mathbb{R}^{n\times n}$ is symmetric, can we find an orthogonal Q so that

$$Q^TAQ = \frac{\mu}{n} I_n + E$$

where $\mu = \text{tr}(A)$ and E has zeros along its diagonal? I think I have an $O(n^2)$ method that computes Q as a product of permutations and Jacobi rotations.

Complete the following function so that it performs as specified

```
  function [Q,B] = ConstantDiag(A)
% A is an n-by-n symmetric matrix with trace mu
% Q is orthogonal such that Q'*A*Q = B with B(i,i) = mu/n for i=1:n
```

Justify your method and email a copy of your implementation. With the rest of the assignment, submit verifications of all the "why?" questions above.