CS 621: Final Exam

December 18, 2002

3:00-5:30pm

Initial each of the following six (6) pages. For the sake of partial credit, you must show your work and comment your solutions as appropriate. If you need more space, write on the reverse side.

____________________ (Print Name)	1. ________ (15 pts)
	2. ________ (15 pts)
	3. ________ (15 pts)
____________________ (Sign Name)	4. ________ (20 pts)
	5. ________ (20 pts)
____________ (ID)	6. ________ (15 pts)

1. This problem is about solving lower bidiagonal least squares problems, e.g.,

$$\min \left\| \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \\ 0 & 0 & a_{43} \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \right\|_2$$

Assume the availability of the function

```
   function [c,s] = Givens(x,y)
% x and y are real scalars
% c and s are real scalars with the property that c^2 + s^2 = 1 and
% sx + cy = 0
```

Complete the following function so that it performs as specified

```
   function xLS = LBiDiagLS(A,b)
% A is (n+1)-by-n and b is (n+1)-by-1
% A(i,j) = 0 if i>j+1 or i<j
% A(j+1,j) is nonzero for j=1:n
% xLS is a columnn-vector that minimizes norm(Ax - b,2)
```

Your implementation must compute the QR factorization of A and make effective use of `Givens`. You are not allowed to invoke the \ operator or the MATLAB `qr` function.

2. A symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$ can be factored $A = LDL^T$ where L is unit lower triangular and D is diagonal. If $v \in \mathbb{R}^n$ then $A(t) = A + tvv^T$ is positive definite for all $t \geq 0$. Suppose $A(t) = L(t)D(t)L(t)^T$ is its LDL^T factorization. In this problem you must show how to compute the diagonal of $\dot{D}(0)$ assuming the availability of

```
   function [L,D] = LDLT(A)
 % A is symmetric positive definite. L is unit lower triangular and
 % D is diagonal so that A = LDL'
```

In particular, complete the following function so that it performs as specified:

```
  function d = dDiag(A,v)
% A is an n-by-n symmetric positive definite matrix and v is n-by-1.
% Let A(t) = A + t*v*v' have L-D-L^transpose factorization
%
%        A(t) = L(t)*D(t)*L(t)'
%
% d is a column n-vector so d = diag( d/dt D(t)) evaluated at t = 0.
```

You may use the \ operator. Hint: Differentiate both sides of $A + tvv^T = LDL^T$ with respect to t and set $t = 0$.

3. For $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ define the affine space

$$S(A,b) = \{y \mid y = Ax + b,\ x \in \mathbb{R}^n\}$$

In this problem you are to show how to compute the minimum value of $\| y_1 - y_2 \|_2$ where $y_i \in S(A_i, b_i)$, $i = 1{:}2$. In particular, complete the following function so that it performs as specified.

```
  function delta = Sep(A1,b1,A2,b2)
% A1 is m-by-n1 and b1 is m-by-1
% A2 is m-by-n2 and b2 is m-by-1
% Assume (n1 + n2) < m
% delta is the smallest value of norm(y1-y2,2) where
% y1-b1 is in the range of A1 and y2-b2 is in the range of A2.
```

You are allowed to use the \ operator.

4. In the least squares problem with linear equality constraints we are asked to minimize $\| Ax - b \|_2$ subject to the constraint that $Cx = d$. If $A \in \mathbb{R}^{m\times n}$ has rank n and $C \in \mathbb{R}^{p\times n}$ has rank p, then the QR factorization can be used:

```
  function x = LSE(A,b,C,d)
% A is m-by-n with rank(A) = n
% b is m-by-1
% C is p-by-n with rank(C) = p
% d is p-by-1
% x is n-by-1 and minimizes norm(Ax - b) such that Cx = d.

[p,n] = size(C); [Q,R] = qr(C');

% New problem: min norm( [ A*Q1 A*Q2][y1;y2] - b) such that [R(1:p,1:p)' 0][y1;y2] =
% where Q1 = Q(:,1:p) and Q2 = Q(:,p+1:n) and [y1;y2] = [Q1 Q2]'*x

y1 = R(1:p,1:p)'\d;
y2 = (A*Q(:,p+1:n))\(b- A*(Q(:,1:p)*y1));
x = Q*[y1;y2];
```

Modify `LSE` so that it can handle the case when C is rank deficient. In particular, complete the following function so that it performs as specified.

```
  function x = LSEG(A,b,C,d)
% A is m-by-n with rank(A) = n
% b is m-by-1
% C is p-by-n with rank(C) <=  p
% d is p-by-1 and in the range of C.
% x is n-by-1 and minimizes norm(Ax - b,2) subject to the constraint that
% norm(Cx - d,2) is minimized.
```

Make effective use of the MATLAB `svd` function. Declare a computed singular value to be zero if it is less that $10^{-12}\hat{\sigma}_1$ where $\hat{\sigma}_1$ is the largest computed singular value. You are allowed to solve full-rank least squares problems using \.

5. Assume that $X, Y \in \mathbb{R}^{n\times r}$ and that $r << n$. The matrix $C = XY^T - YX^T$ is skew-symmetric. Assume the availability of

```
  function [Q,D] = SkewSchur(C)
% C an n-by-n skew symmetric matrix
% Q is orthogonal and D is block diagonal with 1-by-1 and 2-by-2 diagonal blocks
% so Q'C*Q = D.
```

Implement the following function so that it performs as specified

```
  function [Q,D] = SpecSkewSchur(X,Y)
% X and Y are n-by-r matrices and r << n.
% Q is orthogonal and D is block diagonal with 1-by-1 and 2-by-2 diagonal blocks
% so Q'*(eye(n,n) + X*Y' - Y*X')*Q = D.
```

Hint:

```
  function [Q,D] = SpecialSchur(X,B)
% X is an n-by-p matrix, 1<=p<n
% B is a symmetric p-by-p matrix
% A = I + X*B*X'
% Q'*A*Q = D is the Schur decomposition of A.

[n,p] = size(X);
[Q,R] = qr(X);
R1 = R(1:p,1:p);
% Q'(I + XBX')Q = I + [R1*B*R1' 0 ; 0 0]
[Q1,D1] = schur(R1*B*R1');
Q(:,1:p) = Q(:,1:p)*Q1;
D = [eye(p,p)+D1 zeros(p,n-p); zeros(n-p,p) eye(n-p,n-p)];
```

6. Suppose the real matrix

$$T = \begin{bmatrix} T_{11} & v & w \\ 0 & \lambda & \beta \\ 0 & \alpha & \lambda \end{bmatrix} \qquad T_{11} \in \mathbb{R}^{m \times m},\ v, w \in \mathbb{R}^m,\ \alpha < 0,\ \beta > 0$$

has distinct eigenvalues. Assume that T_{11} is upper triangular. Develop a back-substitution process for computing a nonzero vector $x \in \mathbb{C}^{m+2}$ such that

$$Tx = \left(\lambda + i\sqrt{-\alpha\beta} \right) x$$

Express your answer in the form of a MATLAB script assuming that `T` and `m` are available. You are allowed to use \ but no other MATLAB function.