CS 621: Assignment 5

Due: Monday, November 25, , 2002 (In Lecture or 4130 Upson by 4pm))

Scoring on each problem is on a 0-1-2-3 scale. 3 = complete success, 2 = overlooked a small point, 1 = germ of the right idea, 0 = missed the point of the problem. One point will be deducted for insufficiently commented code. Test drivers and related material will be posted on the course website. For each problem submit output and a listing of all the scripts/functions that you had to write/modify in order to produce the output. You are allowed to discuss *background* issues with other students, but the codes you submit must be your own.

Problem 1. If $A \in \mathbb{R}^{n\times n}$ is skew-symmetric then it has a real Schur decomposition of the form $Q^TAQ = D$ where Q is orthogonal and D is block diagonal with 1-by-1 and 2-by-2 diagonal blocks. Making effective use of the MATLAB function `schur`, implement the following:

```
  function [Q,D] = skew34(A)
% A is either a 3-by-3 or 4-by-4 skew-symmetric matrix with distinct eigenvalues
% Q'*A*Q = D is its real schur form with Q chosen so that norm(Q-I,'fro')
% is minimized.
```

Test your implementation on the script `P1.m` that is available on the website. Hint: If A has distinct eigenvalues and $Q_1^TAQ_1 = D_1$ and $Q_2^TAQ_2 = D_2$ are two real Schur decompositions, then $Q_2 = Q_1 \mathrm{d}iag(\pm 1)P$ where P is a permutation.

Problem 2. Suppose $A \in \mathbb{R}^{n\times n}$ is skew symmetric and partition it as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & \cdots & \cdots & A_{mm} \end{bmatrix}$$

If $n = 2m$ then $A_{ij} \in \mathbb{R}^{2\times 2}$, all i and j. If $n = 2m-1$, then $A_{im} \in \mathbb{R}^{2\times 1}$ for $i = 1{:}m-1$, $A_{m,j} \in \mathbb{R}^{1\times 2}$ for $j = 1{:}m-1$, $A_{mm} \in \mathbb{R}^{1\times 1}$, and $A_{ij} \in \mathbb{R}^{2\times 2}$ otherwise. Define

$$BlockOff(A) = \sqrt{2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \| A_{ij} \|_F^2}$$

and appreciate this as a measure of A's departure from block diagonal form. Formulate a cyclic-by-block-row Jacobi process that computes the real Schur decomposition of a skew-symmetric matrix. In the (i,j) subproblem the real Schur decomposition of

$$B = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{bmatrix}$$

must be computed. Following these ideas, implement the following function:

```
  function [Q,D,SweepHistory] = SkewJacobi(A,SmallSkew,tol,itmax)
% A is an n-by-n skew-symmetric matrix
% SmallSkew is a string that names a function f with the
%      property that if B is a 3-by-3 or 4-by-4 skew symmetric
%      matrix then [U,S] = feval(SmallSkew,B) and U'*B*U = S is B's real
%      Schur decomposition.
% tol is a positive real number
% itmax is a positive integer
%
```

```
% Let Ak be the A-matrix after k cyclic-by-block row sweeps have been
% performed, i.e., Ak = Qk'*A*Qk where Qk is orthogonal.
%
% SweepHistory is a column p-vector with the property
% that SweepHistory(k) = BlockOff(Ak), k=1:p
%
% p = itmax or the smallest p such that BlockOff(Ak) <= tol*norm(A,'fro'),
% whichever comes first.
%
% Q = Qp and D = Ap
```

Test your implementation on the script `P2.m` that is available on the website.

Problem 3. Suppose $A = I_n + XBX^T$ where $X \in \mathbb{R}^{n\times p}$ $(p < n)$ and $B \in \mathbb{R}^{p\times p}$ is symmetric. Note that A has at most p eigenvalues that do not equal one. Thus, if $Q^TAQ = T$ is A's tridiagonalization, then T will have a lot of zeros along its subdiagonal. One could exploit this property when getting T's eigenvalues. A better approach starts by computing X's QR factorization. Think about this and implement the following function accordingly.

```
  function [Q,D] = SpecialSchur(X,B)
% X is an n-by-p matrix, 1<=p<n
% B is a symmetric p-by-p matrix
% A = I + X*B*X'
% Q'*A*Q = D is the schur decomposition of A.
```

Test your implementation on the script `P3.m` that is available on the website.

Problem 4. Read §8.7. Note that if $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are symmetric and positive definite, then the generalized eigenvalue problem $Ax = \lambda Bx$ can be approached as a generalized singular value problem $GG^Tx = \lambda KK^Tx$ where G and K are the Cholesky factors of A and B respectively. Using the CS decomposition solution framework outlined on page 467, implement the following function:

```
  function [X,lambda] = GenEval(A,B)
% A and B are n-by-n symmetric positive definite matrices
% X is a nonsingular n-by-n matrix
% lambda is a column n-vector
% For j=1:n, AX(:,j) = lambda(j)*B*X(:,j)
```

Test your implementation on the script `P4.m` that is available on the website.