CS 621: Matrix Computations
Fall 2001
Prelim 2

Handed out: Tues., Nov. 6 or Fri., Nov. 9.

This exam has four questions. The questions are weighted equally. It counts for 20% of your final course grade (same as Prelim 1). **If you pick it up on Tuesday, Nov. 6, then this exam is due back at the end of lecture (12:05) on Friday, Nov. 9. If you pick it up on Friday, Nov. 9, then it is due at the end of lecture (12:05) on Monday, Nov. 12.**

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Golub & Van Loan, Trefethen & Bau and lecture notes, then you must cite them.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until Tuesday, November 13. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any email or otherwise make any on-line posting concerning this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in after lecture but before 5:00 p.m. on the due date will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. Let A be an $n \times n$ nonsingular matrix. What is the minimum possible value of $\|E\|_F$ such that $A + E$ is singular? Justify your answer, which ideally will be in terms of the singular values of A.

2. Let $A \in \mathbf{R}^{n\times n}$ be symmetric and positive definite. The *Cholesky iteration* is given by

$P_0 = A$;
for $k = 0, 1, 2, \ldots$
    Factor $P_k = R_k^T R_k$ where R_k is $n \times n$ upper triangular;
        (Cholesky factorization)
    Let $P_{k+1} = R_k R_k^T$;
end

(a) Show (by induction) that each P_k is symmetric and positive definite, so that the first step of the inner loop makes sense.

(b) Show that each $P^{(k)}$ is similar to (i.e., has the same eigenvalues as) A. Hint: Define $G_k = R_0^T \cdots R_k^T$. Show that $P_k = G_{k-1}^{-1} A G_{k-1}$.

(c) Show that this method actually has a power method hidden inside of it. In particular, argue that $G_{k+1}(:,1)$ is equal to $AG_k(:,1)$ up to a scalar multiple.

3. In this question, we consider the implementation of the Cholesky iteration in the previous question in the special case of bidiagonal matrices. (Note: you may do this question and preceding question in any order; there is no direct relation between their solutions).

   Let $B \in \mathbf{R}^{n\times n}$ be an upper bidiagonal matrix. "Upper bidiagonal" means $B(i,j) = 0$ whenever $i > j$ or $i < j+1$. Let $T = BB^T$ and let $\hat{B}$ be the Cholesky factor of T, that is, an upper triangular matrix such that $T = \hat{B}^T\hat{B}$.

   (a) Show that $\hat{B}$ is also bidiagonal. To ease the grading task, please use the following notation. Let the diagonal entries of B be $\alpha_1, \ldots, \alpha_n$ and the superdiagonal entries be $\beta_1, \ldots, \beta_{n-1}$. Let the diagonal entries of $\hat{B}$ be $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ and superdiagonal entries be $\hat{\beta}_1, \ldots, \hat{\beta}_{n-1}$.

   (b) Propose an algorithm that computes the entries of $\hat{B}$ directly from the entries of B using a recurrence relationship, that is, formulas that define $\hat{\alpha}_i$ and $\hat{\beta}_i$ in terms of α_i's, β_i's and previously computed values of $\hat{\alpha}_j$'s and $\hat{\beta}_j$'s. Your algorithm should have the following properties: (1) it should be $O(n)$ flops, (2) it should not require any arrays (beyond the storage needed for the α's, β's, $\hat{\alpha}$'s and $\hat{\beta}$'s), (3) it should use only $+, *, /, \sqrt{}$, but not subtraction. For maximum points, try to reduce the number of flops by eliminating redundant computations (but without introducing subtractions). [Hint: As part of your recurrence, introduce a variable, say γ, that stands for $\hat{\alpha}_i^2 - \beta_i^2$ (but compute this variable recurrently without any subtraction).]

   Note: The results of the previous two question are taken from the literature on singular value decomposition. The sources for these result will be given in the solution set.

4. (a) Let $\mathbf{u}, \mathbf{v}$ be nonzero vectors. Show that $\|\mathbf{u}\mathbf{v}^T\|_2 = \|\mathbf{u}\|_2 \cdot \|\mathbf{v}\|_2$. Note: the left-hand side uses the matrix 2-norm, whereas the right-hand side uses the vector 2-norm.

   (b) Use the result of part (a) to demonstrate the following, which revisits Q4 of PS4. Given $A \in \mathbf{R}^{n\times n}$, $\mathbf{b} \in \mathbf{R}^n$, and nonzero $\hat{\mathbf{x}} \in \mathbf{R}^n$, consider finding the minimum E measured in the matrix 2-norm (not the Frobenius norm!) such that $(A+E)\hat{\mathbf{x}} = \mathbf{b}$. Show that the exact optimal solution to this problem is $E = \beta\mathbf{r}\hat{\mathbf{x}}^T$, where $\mathbf{r} = \mathbf{b} - A\hat{\mathbf{x}}$ and β is a scalar that you will determine.