CS 621: Matrix Computations
Fall 2001
Practice Prelim 1

Handed out: Fri., Sep. 21 (on the web).

This was a timed 75-minute closed-book and closed-note exam.

1. [**5 points**] What is the definition of "Cholesky factorization"?

2. [**10 points**] In class it was stated that finding roots of the quadratic polynomial $p(x) = x^2 - 2x + 1$ is ill-posed. Suppose that this same problem instance is given, but as part of the problem statement it is known in advance that $p(x)$ has a double root. Argue that this modified problem is well conditioned. In other words, argue that nearby quadratic equations that also have double roots must have nearby solutions. [Hint: If $p(x)$ has a double root r, then r is also a root of $p'(x)$.] Your analysis may drop high-order terms.

3. [**15 points**] In a *bidiagonal* upper triangular matrix U, $U(i,j)$ can be nonzero only if $j = i$ or $j = i + 1$. In other words, the only nonzero entries in U are on the main diagonal and on the immediately adjacent superdiagonal. Write down a specialized efficient algorithm for back-substitution on a bidiagonal upper triangular matrix. Use Matlab notation. Analyze the number of flops in your algorithm (accurate to the leading term).

4. [**15 points**] Let $A \in \mathbf{R}^{n\times n}$ be nonsingular and let $\mathbf{x}, \mathbf{b} \in \mathbf{R}^n$ satisfy $A\mathbf{x} = \mathbf{b}$. Suppose $\hat{\mathbf{x}} \in \mathbf{R}^n$ is some other vector such that $\|A\hat{\mathbf{x}} - \mathbf{b}\| = \delta$. Derive an upper and lower bound on $\|\mathbf{x} - \hat{\mathbf{x}}\|$ in terms of δ. Your bounds may also refer to $\|A\|$ and $\|A^{-1}\|$. Assume the matrix norm is induced by the vector norm.

5. [**15 points**] Let $A \in \mathbf{R}^{(2n)\times(2n)}$ be the block matrix

$$A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

   in which each block is $n \times n$. Suppose plain Gaussian elimination (no pivoting) is applied to A, modified to take advantage of the special structure. How many flops, accurate to the leading term, are required? Is possible to take advantage of the special structure if one uses GEPP? How many flops (accurate to the leading term) are required for GEPP applied to this matrix?

6. [**15 points**] Show that for any $A \in \mathbf{R}^{n\times n}$, $\text{rank}(A) = \text{rank}(A^TA)$. [Hint: argue that the following are equivalent (a) $A\mathbf{x} = \mathbf{0}$; (b) $A^TA\mathbf{x} = \mathbf{0}$; (c) $\mathbf{x}^TA^TA\mathbf{x} = 0$ because (a) implies (b); (b) implies (c); (c) implies (a).]