CS 621: Matrix Computations
Fall 2001
Final Exam

Handed out: Fri., Dec. 7 or Mon., Dec. 10.

This exam has four questions. The questions are weighted equally. It counts for 30% of your final course grade. **If you pick it up on Friday, Dec. 7, then this exam is due back in 493 Rhodes at 12:05 on Monday, Dec. 10. If you pick it up on Mon, Dec. 10, then it is due back in 493 Rhodes at 12:05 on Thursday, Dec. 13.**

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Golub & Van Loan, Trefethen & Bau and lecture notes, then you must cite them.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until Friday, December 14. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any email or otherwise make any on-line posting concerning this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in after 12:05 but before 5:00 p.m. on the due date will still be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. Let $T \in \mathbf{R}^{n\times n}$ be tridiagonal and symmetric positive definite.

   (a) Show that T can be factored as UDU^T where U is unit upper triangular and bidiagonal and D is diagonal. Indeed, give an $O(n)$-flop recurrence relationship to compute the entries of D and U. [Note: The recurrence should iterate backward instead of forward.]

   (b) Show that your recurrence from (a) cannot fail (i.e., cannot divide by zero). Use the assumption that T is positive definite in your argument.

   (c) Let T also be factored as $L\bar{D}L^T$ as in lecture, where L is unit lower triangular and bidiagonal and $\bar{D}$ is diagonal. Show that the (k,k) entry of T^{-1} can be expressed with a very simple formula involving $D(k,k)$, $\bar{D}(k,k)$, and $T(k,k)$. [Hint: Let $\mathbf{z}$ be the kth column of T^{-1}. The following three equations are all true (why?): $T\mathbf{z} = \mathbf{e}_k$, $L^T\mathbf{z} = \bar{D}^{-1}L^{-1}\mathbf{e}_k$, $U^T\mathbf{z} = \bar{D}^{-1}U^{-1}\mathbf{e}_k$. Write out (in terms of individual matrix entries) the kth equation for each of these three equations, and then try to eliminate factors not subscripted by k.]

For both parts, please use the following notation. Let the diagonal of T be $\alpha_1, \ldots, \alpha_n$ and the off-diagonal be $\beta_1, \ldots, \beta_{n-1}$. Let the off-diagonal of L be $g_1, \ldots, g_{n-1}$ and of U be $h_1, \ldots, h_{n-1}$. Finally, let the diagonal of D be $d_1 \ldots, d_n$ and of $\bar{D}$ be $\bar{d}_1, \ldots, \bar{d}_n$.

Note: the literature source for this question will be posted on the course webpage after this exam is over. The point of this question is that the algorithm suggested by question 1 of PS6 can be efficiently implemented.

2. Let $A \in \mathbf{R}^{n\times n}$ be a symmetric positive semidefinite matrix, and let $\mathbf{b}$ be a vector lying in $\mathbf{R}^n$. Assume also that $\mathbf{b} \in \text{Range}(A)$. Show that the conjugate gradient method using starting guess $\mathbf{x}_0 = \mathbf{0}$ will still converge exactly to a true solution of $A\mathbf{x} = \mathbf{b}$ in at most n steps. Assume exact arithmetic. Under the hypotheses of the question, there could be could be more than one solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. Therefore, comment on which solution the method will converge to.

   [Hint: First solve this in the special case that A is diagonal. Then use diagonalization to transform the general case to the special case.]

3. Let $A \in \mathbf{C}^{n\times n}$ be arbitrary, and let $M(A) \subset \mathbf{C}$ be defined by

$$M(A) = \{\mathbf{q}^H A\mathbf{q} : \mathbf{q} \in \mathbf{C}^n, \|\mathbf{q}\|_2 = 1\}.$$

   (a) In the case A is Hermitian, show that $M(A)$ is an interval of the real line given by $M(A) = [\lambda_{\min}(A), \lambda_{\max}(A)]$.

   (b) In the general case, show that all eigenvalues of A are contained in $M(A)$.

   (c) Also in the general case, show that all Ritz values of A produced by Arnoldi for any starting vector (assuming no breakdown) lie in $M(A)$.

4. Let $T \in \mathbf{C}^{n\times n}$ be an upper triangular matrix with a zero on the diagonal, say in position (k, k). The problem is to find a $\mathbf{x} \in \mathbf{C}^n$ such that $T\mathbf{x} = \mathbf{0}$ and $\|\mathbf{x}\|_2 = 1$. This problem was considered as Q1 on PS5 under some restrictive assumptions. Propose an algorithm that will be work well even if there are repeated zeros or near-zeros on the diagonal of T, without worrying about overflow or divide by zero. Analyze the number of complex flops (accurate to the leading term) required by your algorithm to obtain $\mathbf{x}$. (The number of flops depends on k but not n; why?)

   [Hint: The problem is trivial if $k = 1$. Why? For the case $k > 1$, apply Givens rotations on the right of T to zero out diagonal entries and eventually transform this problem to the trivial case.]