Constructive Analysis in Nuprl

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Preliminaries

Use $\mathbb{S}_{k,T} = \mathbb{N}_k \rightarrow T$ and $\mathbb{S}_T = \mathbb{N} \rightarrow T$ for finite and infinite sequences of an arbitrary type T.

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$$s_1 +\!\!+ s_2 = \lambda i.\ \text{if } i < k \text{ then } s_1(i) \text{ else } s_2(i-k)$$

And we use $s.t$ for $s +\!\!+ \overline{t}$, where $\overline{t} = \lambda i.\ t$.

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A relation R of type $n{:}\mathbb{N} \to \mathbb{S}_{n,T} \to \mathbb{P}$ is the same as proposition about the members of the *spread* $n{:}\mathbb{N} \times \mathbb{S}_{n,T}$.

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$$\forall p{:}\mathbb{S}_T.\ \downarrow\exists k{:}\mathbb{N}.\ B(n,p)$$

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- Bar Induction for decidable bars (BID) is the principle that says if R is inductive, and B is a decidable bar, and $\forall n{:}\mathbb{N}.\forall s{:}\mathbb{S}_{T,n}.\ B(n,s) \Rightarrow R(n,s)$ then $R(0, \text{null})$

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Suppose $i : R$ is inductive, and $d : B$ is decidable, and $\_ : B$ is a bar, and $f : \forall n{:}\mathbb{N}.\forall s{:}\mathbb{S}_{T,n}.\ B(n,s) \Rightarrow R(n,s)$

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Suppose $i : R$ is inductive, and $d : B$ is decidable, and $\_ : B$ is a bar, and $f : \forall n{:}\mathbb{N}.\forall s{:}\mathbb{S}_{T,n}.\ B(n,s) \Rightarrow R(n,s)$
Claim $\text{BR}(d,f,i,0,\text{null}) \in R(0,\text{null})$.
Where BR is recursively defined by:

$$\begin{aligned}\text{BR}(d,f,i,n,s) = {} & \text{let } c = d(n,s) \text{ in} \\ & \text{if isl}(c) \text{ then } f(n,s,\text{outl}(c)) \\ & \text{else } i(n,s,\lambda t.\ \text{BR}(d,f,i,n+1,s.t))\end{aligned}$$

“Bootstapping” Bar Induction

Write BR(n, s) for BR(d, i, f, n, s). Then

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Let

$$Q(n,s) = (\mathrm{BR}(n,s) \in R(n,s))$$

If $d(n,s) = \mathrm{inl}(b)$ for $b \in B(n,s)$ then $\mathrm{BR}(n,s) = f(n,s,b) \in R(n,s)$. Therefore, $B(n,s) \Rightarrow Q(n,s)$.

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Theorem

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Proof.

For $s \in \mathbb{S}_{T,n}$ and $p \in \mathbb{N} \rightarrow T$ write $s \prec p$ for $p = s \in \mathbb{S}_{T,n}$. Let $Q(n, s)$ be the proposition

$$\exists b{:}\mathbb{N}.\ \forall p{:}\mathbb{N} \rightarrow T.\ (s \prec p) \Rightarrow \exists k{:}\mathbb{N}_b.\ A(n+k, p)$$

If Q is inductive, then by BID, $Q(0, \text{null})$, which says that A is a uniform bar.

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If Q is inductive, then by BID, $Q(0, \text{null})$, which says that A is a uniform bar. Suppose that $\forall t{:}T.\ Q(n + 1, s.t)$ then we have a function $f \in T \to \mathbb{N}$ such that $s.t \prec p \Rightarrow \exists k{:}\mathbb{N}_{f(t)}.\ A(n + 1 + k, p)$. Any path p that agrees with s agrees with some $s.t$—so $Q(n, s)$ follows for $b = 1 + \max\{f(t) \mid t \in T\}$ which exists (and is computable) because T is finite. Therefore Q is inductive. □

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- From the continuity principle, we derive $(*)$:

$$\forall F:\mathbb{C} \to \mathbb{Z}.\ \downarrow \exists M:(\mathbb{C} \to \mathbb{N}).\ \forall f, g:\mathbb{C}.\ (g = f \in \mathbb{C}_{M(f)}) \Rightarrow F(g) = F(f)$$

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Lemma

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From $(*)$ we get $M \in \mathbb{C} \to \mathbb{N}$ such that $\forall f,g{:}\mathbb{C}.\ (g = f \in \mathbb{C}_{M(f)}) \Rightarrow F(g) = F(f)$. We also get $X \in \mathbb{C} \to \mathbb{N}$ such that $\forall f,g{:}\mathbb{C}.\ (g = f \in \mathbb{C}_{X(f)}) \Rightarrow M(g) = M(f)$.

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Lemma

(uniform continuity step2) For all $F \in \mathbb{C} \rightarrow \mathbb{Z}$

$$\exists b : \mathbb{N}.\ \forall f, g : \mathbb{C}.\ (g = f \in \mathbb{C}_b) \Rightarrow F(g) = F(f)$$

A Seemingly Impossible Program

Corollary

If $R(n, m)$ is a decidable relation for $n, m \in \mathbb{Z}$ and F is a function of type $\mathbb{C} \rightarrow \mathbb{Z}$ then it decidable whether $\exists f, g : \mathbb{C}.\ R(F(f), F(g))$

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Proof.

By Step 2, there is a $b \in \mathbb{N}$ such that $\forall f, g{:}\mathbb{C}.\ (g = f \in \mathbb{C}_b) \Rightarrow F(g) = F(f)$. So all possible values for $F(f)$ are already present among the $G(s) = F(s +\!\!+_b \overline{0})$ for $s \in \mathbb{C}_b$. There are only a finite number of pairs $\langle s_1, s_2 \rangle$ in $\mathbb{C}_b \times \mathbb{C}_b$ so we can enumerate them and check for each of them whether $R(G(s_1), G(s_2))$. □

Uniform continuity for real functions

Recall that an operation $f \in I \to \mathbb{R}$ is an extensional real function if

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Theorem

(Brouwer's theorem for proper closed intervals) For all real numbers $a < b$ and all operations $f \in [a, b] \to \mathbb{R}$

$$FUN(f, [a, b]) \Leftrightarrow UCONT(f, [a, b])$$

Proof of Brouwer's Theorem for proper closed intervals

The direction $\text{UCONT}(f, [a, b]) \Rightarrow \text{FUN}(f, [a, b])$ is easy because, if $x \equiv y$ then $|x - y| = 0$. The assumption $\text{UCONT}(f, [a, b])$ then implies that $\forall \epsilon > 0.\ |f(x) - f(y)| < \epsilon$ and this implies $f(x) \equiv f(y)$.

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$|f(G(p)) - j/4n| + |f(G(q)) - j/4n| < 1/2n + 1/2n < \epsilon$ because, for any real number x, $|x - x_{2n}/4n| < 1/2n$.

Proof of Brouwer' Theorem, continued

Suppose the map G has the property that, for some function h mapping positive integers to positive real numbers, if $x, y \in [a, b]$ and $|x - y| < h(k)$ then there exists $p, q \in \mathbb{C}$ with $G(p) \equiv x$, $G(q) \equiv y$, and $p = q \in \mathbb{C}_k$.

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We can construct such a G for which $h(k) = (2/3)^k * (b - a)/6$, (but only when $a < b$). (See the notes.)
That gives us Brouwer's Theorem for interval $[a, b]$ when $a < b$.
But we want it when we know only $a \leq b$.

Brouwer's Theorem for all closed intervals

Theorem

(Brouwer's theorem) For all real numbers $a \leq b$ *and all operations* $f \in [a, b] \to \mathbb{R}$

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Otherwise, for all p and q in $\mathbb{C}$,

$$(**) \qquad |f(G(p))_{4n} - f(G(q))_{4n}| \leq 4$$

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Then, $|u - v| \leq |u - u_{4n}/8n| + |v - v_{4n}/8n| + |u_{4n} - v_{4n}|/8n \leq 1/4n + 1/4n + 4/8n = 4/4n = 1/n$.

Simplification of formal theory

For an operation defined on an interval, $f \in I \to \mathbb{R}$, Bishop defines continuity of f to be uniform continuity on all compact (i.e. closed, finite, non-empty) sub-intervals of I. With this definition of continuity, Brouwer's theorem implies that if $\forall x, y\!:\!I.\ x \equiv y \Rightarrow f(x) \equiv f(y)$ then f is continuous on I.

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5. The integral $\int_a^b f(x)dx$ exists.

A functional equation of Cauchy

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