Lecture 17: The Constructive Real Numbers Continued

Mark Bickford and Robert Constable

Abstract

Dr. Mark Bickford continued his discussion of the constructive real numbers. He demonstrated a service he has provided in Nuprl for evaluating expressions that compute with real numbers. He then presented Brouwer's Continuity Principle in Nuprl using the type constructor called *half-squash* written as $\downarrow P$. We have discussed the *full squash* operator previously using the notation $\{Unit|P\}$. This is also written as $\Downarrow P$. [1]

1 Introduction

We have made the case that the constructive real numbers are useful in writing correct programs for controlling robots, self driving cars, and other *cyber physical systems* and that they are a fundamental concept in constructive mathematics just as the real numbers are in classical mathematics. We let $\mathbb{R}$ denote the constructive reals. Achieving a robust and applicable implementation has been a topic of interest in computer science at least since 1991 when the Nuprl system provided the first implementation of them [7]. Since that time the development and application of constructive analysis has been a regular topic of research using the Cornell type theory.

Mark Bickford has advanced this theory of intuitionistic analysis to the point where he is able to formulate and prove very interesting new theorem about the real numbers which he calls a *Connectedness Theorem*. He proves this result in the notes for his lectures. This theorem is a fundamental result in constructive analysis that as far as we know is original to Dr. Bickford and Nuprl.

2 Real Numbers

As we have noted before, a foundational understanding of the real numbers is a topic of broad interest in mathematics, logic, and computer science. Perhaps the most accessible

[1] These are not the exact symbols used in the lecture, but a close approximation that we can easily print.

approach to the classical reals as well is to think about how we might compute with them. We are familiar with the decimal expansions of several important reals, such as π which begins with these digits:

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482...

It is possible to compute as many digits of *pi* as anyone would need to use or want to see. So far at least 12.1 trillion digits of *pi* have been computed. We can easily find ten thousand digits of e on the web, it looks like this:

2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217...

It would not be so much fun to multiply these two approximations together to approximate $\pi \times e$, but we know how to do it. Our first intuitive grasp of the constructive reals might be in terms of these *never ending decimal approximations*. This was an approach that Turing investigated [**?**]. However as Turing learned, this is not a good definition of the computable reals if we want to prove properties of them and build a theory of computable real numbers to support the calculus. Already from work of Brouwer that we have implemented in Nuprl, we know that if we use algorithms to define the sequence of digits, say Turing Machines, we do not arrive at an adequate theory rich enough to implement all of the important and practical concepts of Brouwer's intuitionistically computable real numbers nor rich enough to validate all of Brouwer's discoveries about computing with these numbers.

Attempting to understand the Euclidean plane in terms of points and lines leads to insights about the constructive reals, but it does not easily lead to the fundamental concepts discovered by Brouwer. Relating geometric points to constructive reals has been a source of productive insights about the nature of the real numbers, but the account we have thus far is not fully satisfactory and leaves open some questions that are at least a thousand years old. On the other hand, our implementation of the constructive real numbers is an important tool in our computational investigation of geometry.[2]

There are several paths from geometry to the constructive real numbers. One path originates with Newton and Leibniz, motivated in part by the need to compute. The path to formulating the right axioms on the reals usually starts with the algebraic operations such as adding, subtracting, multiplying, and dividing real numbers. These axioms state that the real numbers have the algebraic properties of a field. The rational numbers also form a field, so this path also requires a property that distinguishes the reals from the rational numbers. One such property is captured in the *completeness axiom*, i.e. that every bounded non-empty subsets of the reals has a least upper bound.

There are other approaches to understanding the real numbers that are more geometric and topological. It is appealing to to formulate these properties in type theory and discover their "computational meaning" in various ways. This is not a closed topic even

[2] We have already noted that such investigations are very subtle and we can be led astray, thinking the Euclid Proposition 2 is not constructively true, when we know that it really is.

though some of the concepts involved have been studied for hundreds of years. Relatively modern mathematicians such as Weyl [10] and Brouwer [9, 8, 6] and Poincaré [9] have written extremely influential studies of the reals. We were very influenced by the pragmatic approach of the American analyst Errett Bishop [4, 3] and his close collaborator Douglas Bridges [5]. However, in 2016 we changed the Nuprl account to match L.E.J. Brouwer's definition of the reals, leading us to a fully intuitionistic account. We did this on pragmatic grounds, the theory is more useful because it provides better results about continuity. We also took this step because it enriches our type theory in ways that have considerable computational value. We reported on these results in LICS 2017 [1, 2].

If there is time in the course, we will explore other paths to the constructive real numbers that reveal other important aspects of this concept and other paths that lead to them. The path from Euclidean geometry traces the actual history of this very important concept. This data type is so important in science generally that we have made available on the Nuprl web page, as a public service, a calculator that is provably correct and allows researchers who need to know the exact real number values in critical computations to access our verified implementation.

References

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