The Constructive Real Numbers

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Abstract

Dr. Mark Bickford will discuss his implementation of the constructive reals in Nuprl, and we will discuss their significance both in practical applications and in the development of type theory. Supplementary material for this lecture is written by Dr. Bickford and included on the course web page for Lectures 16 and 17. He will also discuss his very insightful new result about the connectedness of the reals.

1 Introduction

The constructive real numbers are useful in writing correct programs for controlling robots, self driving cars, and other *cyber physical systems.* Achieving a robust and applicable implementation has been a topic of interest in computer science ever since 1984 when the Nuprl system provided the first implementation of them. Since that time the development and application of constructive analysis has been a regular topic of research using the Cornell type theory.

Mark Bickford has advanced this theory to the point where he is able to formulate and prove very interesting new theorems about the real numbers such as his *Connectedness Theorem* which he will discuss in the second lecture or possibly in a third lecture depending on how the presentation and discussions unfold.

2 Real Numbers

A foundational understanding of the real numbers is a topic of broad interest in mathematics, logic, and computer science. Understanding the Euclidean plane has been a source of productive insights about the nature of the real numbers. An implementation of the constructive real numbers is an important tool in our computational investigation of geometry. There are several paths from geometry to the constructive real numbers. One path originates with Newton and Leibniz, motivated in part by the need to compute. The path to formulating the right axioms on the reals usually starts with the algebraic operations such as adding, subtracting, multiplying, and dividing real numbers. These axioms state that the real numbers have the algebraic properties of a field. The rational numbers also form a field, so this path also requires a property that distinguishes the reals from the rational numbers. One such property is captured in the *completeness axiom*, i.e. that every bounded non-empty subsets of the reals has a least upper bound.

There are other approaches to understanding the real numbers that are more geometric and topological. It is appealing to to formulate these properties in type theory and discover their "computational meaning" in various ways. This is not a closed topic even though some of the concepts involved have been studied for hundreds of years. Relatively modern mathematicians such as Weyl [17] and Brouwer [16, 15, 8] and Poincaré [16] have written extremely influential studies of the reals. We were very influenced by the pragmatic approach of the American analyst Errett Bishop [2, 1, 3] and his close collaborator Douglas Bridges. However, in 2016 we changed the Nuprl account to match L.E.J. Brouwer's definition of the reals, leading us to a fully intuitionistic account. We did this on pragmatic grounds, the theory is more useful because it provides better results about continuity. We also took this step because it enriches our type theory in ways that have considerable computational value. We reported on these results in LICS 2017 [14].

If there is time in the course, we will explore other paths to the constructive real numbers that reveal other important aspects of this concept and other paths that lead to them. The path from Euclidean geometry traces the actual history of this very important concept. This data type is so important in science generally that we have made available on the Nuprl web page, as a public service, a calculator that is provably correct and allows researchers who need to know the exact real number values in critical computations to access our verified implementation.

3 Relevant Literature

Analysis: [1, 2], [3], [17].

Intuitionism: [4, 5, 6, 7, 9, 12], [13, 10, 11].

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