Markov's Principle in Constructive Type Theory

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Abstract

We provide a constructive semantics for Markov's Principle using partial types. We highlight and discuss the elements of this account that are not universally accepted in constructive mathematics.

1 Introduction

Markov's Principle (MP), proposed in 1954, is a signature feature of Russian constructive mathematics. Brouwer had already been developing his version of constructive mathematics years earlier (by 1908), known as *intuitionistic mathematics*, and he did not accept MP.[1] Both Brouwer and Markov thought deeply about the nature of constructive proofs and made major contributions to logic and mathematics. However, they did not agree on Markov's Principle. With the increasing importance and practical value of implemented constructive formal theories, there is new incentive to reconcile these views. In a sufficiently expressive constructive type theory reconciliation is easy as we show here, elaborating insights of two former students with whom I have discussed this topic in detail [7] over the years and drawing on the article by Coquand and Mannaa [1] on the independence of MP in type theory.

One way to resolve the difference is to take advantage of Kleene's notion of *partial recursive functions* [5]. Among such programs is a realizer for Markov's principle, namely an unbounded search program. If we know that this search cannot continue indefinitely, that knowledge can be construed as evidence that the value being sought will eventually be found, even if we do not know how many steps the search for it requires. However, adding MP to type theory changes our expectations about

[1] Brouwer called his mathematics "modern mathematics."

proving computational complexity bounds on algorithms. For computer scientists, who care a great deal about the computational complexity of algorithms [4], this is a significant issue.

1.1 Markov's Principle in Type Theory

Here is a precise logical way to describe MP over the natural numbers $\mathbb{N}$. Suppose that given a specific predicate $P(x)$ on the natural numbers $\mathbb{N}$, we also know constructively that $\forall n : \mathbb{N}.(P(x) \vee \sim P(x))$. Thus we have a computable function dec_P that can decide whether there is evidence for $P(n)$ or for $\sim P(n)$. Having such a function is required evidence for knowing constructively that $\forall n : \mathbb{N}.(P(x) \vee \sim P(x))$.

Given this justification for $\forall n : \mathbb{N}.(P(x) \vee \sim P(x))$, then starting at 0 and testing the natural numbers in order, we use the constructive evidence $\lambda(x.dec_P(x))$ to justify the assertion $\forall n : \mathbb{N}.(P(x) \vee \sim P(x))$. Using this evidence, we can find out in order $P(0) \vee \sim P(0)$, $P(1) \vee \sim P(1)$, $P(2) \vee \sim P(2)$, etc. Since we know $\sim \forall x : \mathbb{N} \sim P(x))$, MP claims that at some natural number, say m we will discover that $\exists n : \mathbb{N}.P(n)$, namely at $P(m)$.

The obvious way to find m is to ask a program to test $P(0), P(1), P(2), ...$ until it finds the first natural number for which we know $P(m)$ using the decision procedure given by dec_P. We might not have in advance any sense for how many numbers we must test, but Markov claims that this does not matter, eventually we will find the first one based on the meager amount of knowledge provided and the algorithm $\lambda(x.dec_P(x))$. If we know the cost of evaluating $dec_(n)$ for each n, then we can compute the total cost of finding m once we have found it, but not before. Putting all this information together, Markov claims that we know his principle:

Markov's Principle:

$$(\forall n : \mathbb{N}.(P(x) \vee \sim P(x))) \Rightarrow (\sim (\forall x : \mathbb{N} \sim P(x))) \Rightarrow (\exists n : \mathbb{N}.P(n))$$

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1.2 Weak Markov's Principle – WMP

Brouwer's objection to Markov's Principle is based on his understanding of the real numbers. Weak Markov's Principle uses the following definition of a pseudo-positive real.

Definition: A real number a in $\mathbb{R}$ is *pseudo positive* if and only if

$$\forall x \in \mathbb{R}.(\sim\sim (0 \leq x) \vee \sim\sim (x < a)).$$

Weak Markov Principle (WMP): Every pseudo positive real is positive.

We can prove WMP from Markov's principle. There are some interesting open questions about WMP and other constructive theories of the reals, but we will not have time to investigate them in this course. Dr. Bickford is familiar with consequences of WMP in constructive analysis. Next week he will lecture about doing real analysis in Nuprl. He might show an application of WMP.

2 Logician's Bag of Tricks

Once logicians created first-order logic and axiomatized theories of mathematics expressible in that language. They discovered a remarkable "bag of tricks" to stun and amaze us. Here are some of the named results of this kind. We won't cover much of this since its ties to type theory are less well developed.

- Non-standard models of Peano Arithmetic (PA), some theorems of arithmetic fail in these models and hence are not first-order. One of the known results of this sort is Goodstein's theorem. On the other hand, in non-standard models of the reals, we can create infinitesimals. This allows us to formalize some results in the theory that Leibnitz used in developing the calculus. We might look at these later in the course.

- Gödel's double negation interpretation of classical logic in intuitionistic logic. This is covered in detail in Kleene's logic textbook. Kolmogorov made the same discovery [6] in his early twenties.

- Gödel's Dialectica interpretation [3] of intuitionistic arithmetic (also called Heyting Arithmetic (HA) into a quantifier free logic.

- Friedman's A-translation [2] that embeds certain classical theorems of arithmetic (PA) into constructive arithmetic (HA). It gives constructive proofs of the termination of recursive functions that are known to be total in Peano Arithmetic (PA).

- Forcing method of Cohen used, in 1963, to prove the independence of the axiom of choice and the continuum hypothesis from ZermeloFraenkel set theory (ZFC).

- Constructive forcing with Kripke models.

References

[1] Thierry Coquand and Bassel Mannaa. The independence of markov's principle in type theory. *CoRR*, abs/1602.04530, 2016.

[2] Harvey Friedman. Classically and intuitionistically provably recursive functions. In D. S. Scott and G. H. Muller, editors, *Higher Set Theory*, volume 699 of *Lecture Notes in Mathematics*, pages 21–28. Springer-Verlag, 1978.

[3] K. Gödel. Über eine noch nicht benütste erweiterung des finiten standpunktes. *Dialectica*, 12:280–287, 1958.

[4] Juris Hartmanis and R. Stearns. On the computational complexity of algorithms. *Transactions of the American Mathematics Society*, 117:285–306, 1965.

[5] S. C. Kleene. *Introduction to Metamathematics.* D. Van Nostrand, Princeton, 1952.

[6] A. N. Kolmogorov. On the principle '*tertium non datur.*'. *Mathematicheski Sbornik*, 32:646–667, 1924.

[7] Alexei Kopylov and Aleksey Nogin. Markov's principle for propositional type theory. In L. Fribourg, editor, *Computer Science Logic, Proceedings of the 10^{th} Annual Conference of the EACSL*, volume 2142 of *Lecture Notes in Computer Science*, pages 570–584. Springer-Verlag, 2001.