CS6180: Lecture 11

Robert Constable and Ariel Kellison

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1 Lecture Summary

In this lecture we will illustrate the value of constructive type theory in formalizing one of the oldest textbooks in mathematics, Euclid's *Elements* [1, 2, 3]. This is an account of plane geometry as we all know. Heath's book was published in 1909 and is still in print. Many people still read this book. Brouwer was somewhat skeptical that Euclidean geometry could be developed intuitionistically [4]. He was reading the same book. His PhD student Arend Heyting wrote a PhD thesis on projective geometry [**?**] as well as a very accessible book on intuitionistic mathematics [5].

In addition to Heyting's work, we will mention two major books on formalization of Euclidean Geometry, one by David Hilbert [6] and one in German by Alfred Tarski and his students W. Schwabhäuser, and Wanda Szmielew entitled *Metamathematische Methoden in der Geometrie* [7] in first-order logic. Large parts of the German book have been formalized in Coq and some parts also in Nuprl informed by the Coq proofs. Neither the Hilbert nor the Tarski formalizations is constructive, however the Coq effort is largely constructive assuming that equality of points is decidable.

Mark Bickford and Ariel Kellison have formalized in Nuprl elements of the Coq theory. But we do not believe that the Coq formalization uses a realistic assumption about the type of Points. An interesting but challenging project would be to translate some of the Coq proofs into the completely constructive account of Euclid being developed in Nuprl where decidable equality of points is not assumed. Moreover, the Coq proofs are not very readable, and we have made a significant effort to render the Nuprl proofs readable step by step.

We think that the Nuprl formalization is faithful to Euclid and that it draws attention to some of the subtle points about doing geometric constructions using only a *collapsing compass* and a *straight edge*. Our work is also informed by extensive investigations of constructive geometry by Michael Beeson [8, 9, 10, 11, 12]. His accounts are very insightful, but we disagree with his claim that Euclid Proposition 2 is not constructive. This is the key proposition which Euclid uses to show how to simulate a non-collapsing compass with a collapsing one. We do not adopt Beeson's approach to equality in general. Ariel will demonstrate the Nuprl constructive proof of Euclid Proposition 2 and show how Nuprl captures precisely the computational content of Euclid's proofs.

References

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[6] David Hilbert. *Foundations of Geometry.* Open Court Publishing.

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