Constructive Ancestral Logic

As a somewhat more complex example of a constructive interpretation of a logic we here present Ancestral Logic [1]. This is a rather natural extension of first-order logic, obtained by the addition of the transitive closure operator.

To recall, in mathematics, the transitive closure of the binary relation R on X, TC_R , is the smallest transitive relation on X that contains R. An alternative, more constructive, definition is $TC_R = \bigcup_{n\in\mathbb{N}} R^n$ where R^n is defined by $R^0 = R$ and $R^n = R^{n-1} \circ R$ for $n > 0$.

Ancestral logic is defined to be the extension of FOL obtained by the addition of formulas of the form $(TC_{x,y}\varphi)(u,v)$ for any formula φ, x, y distinct variables. The free occurrences of x and y in φ become bound in this formula. The intended meaning of $(TC_{x,y}\varphi)(u,v)$ is that s and t stand in the transitive closure of the binary relation that φ defines on x and y. That is, intuitively, that $(TC_{x,y}\varphi)(u,v)$ is equivalent to the "infinite disjunction":

$$\varphi(u,v) \vee \exists w_1 (\varphi(u,w_1) \wedge \varphi(w_1,v)) \vee \exists w_1\exists w_2 (\varphi(u,w_1) \wedge \varphi(w_1,w_2) \wedge \varphi(w_2,u)) \vee \dots$$

What is the evidence for a TC-formula?

To constructively know $(TC_{x,y}\varphi)(u,v)$, we construct a *list* of elements, say $[a_0, ..., a_n]$, and a list of evidence terms $[r_0, ..., r_{n+1}]$ such that r_0 is evidence for $\varphi(u, a_0)$ and r_{n+1} is evidence for $R(a_n, v)$ and the intermediate terms form an evidence chain, i.e. a_i is evidence for $\varphi(a_{i-1}, a_i)$ for $0 < i \leq n$. Therefore, formally we take the evidence type for $(TC_{x,y}\varphi)(u,v)$ to consist of lists of the form

$$[\langle u, a_0, r_0\rangle, \langle a_0, a_1, r_1\rangle, ..., \langle a_n, v, r_{n+1}\rangle]$$

where the above-mentioned conditions hold.

Proof System

The proof system for Ancestral logic is obtained by the addition of the followings to the system for FOL:

1. $\varphi(u,v) \Rightarrow (TC_{x,y}\varphi)(u,v)$
2. $(TC_{x,y}\varphi)(u,v) \,\&\, (TC_{x,y}\varphi)(v,w) \Rightarrow (TC_{x,y}\varphi)(u,w)$t
3. $(\psi(u,v) \,\&\psi(v,w) \Rightarrow \psi(u,w)) \,\&\, (\varphi(x,y) \Rightarrow \psi(x,y)) \Rightarrow ((TC_{x,y}\varphi)(u,v) \Rightarrow \psi(u,v))$

In the case of number theory, instead of Axiom 13 (the induction principle of PA and HA) it suffices to take $v = 0 \vee (TC_{x,y}y = x')(0,v)$ as an additional axiom. This is because the third TC-axiom is a generalized induction rule that allows for the derivation of arithmetical induction.

How can we derive Axiom 13 in the TC system?

Take $\varphi(x,y) := y = x'$ and $\psi(x,y) := A(x) \Rightarrow A(y)$. The first conjunct of the third TC-axiom is of course true. The second one is true due to the assumption $\forall x.A(x) \Rightarrow A(x')$. Thus, we have $(TC_{x,y}y = x')(u,v) \Rightarrow (A(u) \Rightarrow A(v))$. Substituting 0 for u we get $(TC_{x,y}y = x')(0,v) \Rightarrow (A(0) \Rightarrow A(v))$, from which it is straightforward to derive $(TC_{x,y}y = x')(0,v) \Rightarrow A(v)$, by the assumption $A(0)$. Using the same assumption we get that $v = 0 \Rightarrow A(v)$. Hence, we obtain $v = 0 \vee (TC_{x,y}y = x')(0,v) \Rightarrow A(v)$. Using the additional axiom we are then able to derive $A(v)$.

What should be the realizers for the TC axioms?

1. a list with one element (a triple).
2. a concatenation of the two lists in the hypothesis.
3. Suppose $\psi(u,v) \& \psi(v,w) \Rightarrow \psi(u,w)$ is realized by the function f and $\varphi(x,y) \Rightarrow \psi(x,y)$ by g. The intuitive computation behind this generalized induction principle is recursively computing on the list that realizes $(TC_{x,y}\varphi)(u,v)$, call it r, in the following way: we start with the first two triples, applying g to the third element in both. This results in a chin of two realizers for ψ who can now be combined into one using f. We now move to the next element, first using g to convert the φ-realizer to a ψ-realizer, then using f to combine it with the one created in the previous step. We proceed with this process until eventually we obtain a realizer for $\psi(u,v)$.

Fun fact

Using the transitive closure operator the (constructive) existential quantifier can be defined. How?

$$\exists x\varphi \Longleftrightarrow \left(TC_{a,b}\left(\varphi\left\{\frac{a}{x}\right\} \vee \varphi\left\{\frac{b}{x}\right\}\right)\right)(0,0)$$

(0 in this formula can be replaced by any constant symbol.)

Task: First, convince yourself that this indeed holds. Then, try to write the realizers for both directions of the claim.

References

[1] L. Cohen and R. L. Constable. Intuitionistic ancestral logic. *Journal of Logic and Computation*, 2015.