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┆  $\forall x : \mathbb{N}. (\exists r : \{\mathbb{N} \mid (((r * r) \leq x) \wedge x < (r + 1) * (r + 1))\})$ 
|
BY ((GeneralInductionOnNat THENA Auto) THEN CaseNat 0 'x')
| \
| 1.  $x : \mathbb{N}$ 
| 2.  $\forall x1 : \mathbb{N}x. (\exists r : \{\mathbb{N} \mid (((r * r) \leq x1) \wedge x1 < (r + 1) * (r + 1))\})$ 
| 3.  $x = 0$ 
|  $\vdash \exists r : \{\mathbb{N} \mid (((r * r) \leq 0) \wedge 0 < (r + 1) * (r + 1))\}$ 
| |
1 BY (With  $\lceil 0 \rceil$  (D 0). THEN Auto')
| \
| 1.  $x : \mathbb{N}$ 
| 2.  $\forall x1 : \mathbb{N}x. (\exists r : \{\mathbb{N} \mid (((r * r) \leq x1) \wedge x1 < (r + 1) * (r + 1))\})$ 
| 3.  $\neg(x = 0)$ 
|  $\vdash \exists r : \{\mathbb{N} \mid (((r * r) \leq x) \wedge x < (r + 1) * (r + 1))\}$ 
|
BY ((InstHyp  $\lceil x - 1 \rceil$  2. THENA Auto) THEN D (-1))
|
| 4.  $r : \mathbb{N}$ 
| [5].  $((r * r) \leq (x - 1)) \wedge x - 1 < (r + 1) * (r + 1)$ 
|  $\vdash \exists r : \{\mathbb{N} \mid (((r * r) \leq x) \wedge x < (r + 1) * (r + 1))\}$ 
|
BY ((Evaluate  $\lceil r2 = r \rceil$ . THENA Auto)
| THEN (Evaluate  $\lceil r3 = (r2 + 1) \rceil$ . THENA Auto)
| THEN (Decide  $\lceil x < r3 * r3 \rceil$ . THENA Auto))
| \
| 6.  $r2 : \mathbb{Z}$ 
| 7.  $r2 = r$ 
| 8.  $r3 : \mathbb{Z}$ 
| 9.  $r3 = (r2 + 1)$ 
| 10.  $x < r3 * r3$ 
|  $\vdash \exists r : \{\mathbb{N} \mid (((r * r) \leq x) \wedge x < (r + 1) * (r + 1))\}$ 
| |
1 BY (With  $\lceil r2 \rceil$  (D 0). THEN Auto')
| |
| 5.  $(r * r) \leq (x - 1)$ 
| 6.  $x - 1 < (r + 1) * (r + 1)$ 
| 7.  $r2 : \mathbb{Z}$ 
| 8.  $r2 = r$ 
| 9.  $r3 : \mathbb{Z}$ 
| 10.  $r3 = (r2 + 1)$ 
| 11.  $x < r3 * r3$ 
|  $\vdash (r2 * r2) \leq x$ 
| |
1 BY (ElimVar 'r3' THEN ElimVar 'r2' THEN Auto')
| \
| 6.  $r2 : \mathbb{Z}$ 
| 7.  $r2 = r$ 
| 8.  $r3 : \mathbb{Z}$ 
| 9.  $r3 = (r2 + 1)$ 
| 10.  $\neg x < r3 * r3$ 
|  $\vdash \exists r : \{\mathbb{N} \mid (((r * r) \leq x) \wedge x < (r + 1) * (r + 1))\}$ 
|
BY (With  $\lceil r3 \rceil$  (D 0). THEN Auto')
|
| 5.  $(r * r) \leq (x - 1)$ 

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6. x - 1 < (r + 1) * (r + 1)
7. r2: ℤ
8. r2 = r
9. r3: ℤ
10. r3 = (r2 + 1)
11. ¬x < r3 * r3
12. (r3 * r3) ≤ x
⊢ x < (r3 + 1) * (r3 + 1)
|
BY (ElimVar 'r3' THEN ElimVar 'r2' THEN Auto').
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Extract:

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λx.letrec sqrt(x) =
  if x = 0 then 0
  else let r2 := sqrt (x - 1) in
        let r3 := r2 + 1 in
          if (x) < (r3 * r3) then r2
          else r3 in
  sqrt(x)
```