CS6180 Lecture 21

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1 Topics

(1) Gödel's interpretation of classical arithmetic (PA) into intuitionistic arithmetic (HA).

(2) Defining a hybrid constructive/classical first-order logic using virtual evidence semantics.

(3) Limited applications to constructive non-standard analysis, e.g. constructive infinitesimals and proving continuity constructively using infinitesimals.

2 Hybrid First-Order Logic

Gödel studied the relationship between intuitionistic and classical number theory [2] starting already in 1932, right after his famous incompleteness result. He remained interested in the relationship between classical logic and intuitionistic logic for most of his life. He published another fundamental paper on this topic. His results are "proof theoretic," showing how to convert classical proofs to constructive ones. We are interested in a semantic approach that would let us integrate the two logics. This can be achieved using *virtual evidence* [1]. However, we will look briefly at Gödel's proof theoretic approach first since it is easy to grasp.

The lecture presentation is based on material from Kleene's *Introduction to Metamathematics* [3]. That material is included in the supplementary readings.

We have discussed adding a virtual version of the law of excluded middle. Our syntax uses the type theory set type, $\{P \ \vee \ \sim P\}$ to express a classical version of $P \ \vee \ \sim P$. The type theory definition uses the Unit type which has exactly one element, $\star$. So the type is actually $\{Unit|P \ \vee \ \sim P\}$.

When we introduce virtual evidence into a proof, we must keep track of how the logical rules propagate it. Let us look at a simple example of the method in operation. Consider the following classical propositional fact.

$$\sim (P\&Q) \Rightarrow (\sim P \vee \sim Q).$$

3 Notes on Infinitesimals and Hyperreal Numbers, $\mathbb{R}^*$

$0 < x < \frac{1}{n}$	$n > 0$	x is a positive infinitesimal
$\frac{-1}{n} < x < 0$	$n > 0$	x is a negative infinitesimal

I. Axioms for Infinitesimals

(a) There is a positive infinitesimal.

(b) If ϵ is a positive infinitesimal then $-\epsilon$ is a negative infinitesimal, and $r + \epsilon$ is hyperreal but not real.

(c) If $\epsilon > 0$ and $a > 0$, then $a \cdot \epsilon$ is a positive infinitesimal.

(d) If $\epsilon > 0$ then $\frac{1}{\epsilon}$ is a positive infinitesimal and $\frac{-1}{\epsilon}$ is a negative infinitesimal.

II. Algebraic Axioms for Hyperreals, $\mathbb{R}^*$

(a) $\forall x : \mathbb{R}.\ x \in \mathbb{R}^*$

(b) $\forall x, y : \mathbb{R}^*.\ x + y,\ x \cdot y,\ x - y$ belong to $\mathbb{R}^*$

(c) $\forall x : \mathbb{R}^*.\ x \neq 0 \Rightarrow \frac{1}{x} \in \mathbb{R}^*$

(d) The associative, commutative, distributive, and identity laws for $\mathbb{R}$ hold for $\mathbb{R}^*$.

III. Order Axioms for $\mathbb{R}^*$

1.	$a < b \ \& \ b < c \Rightarrow a < c$	(transitivity)
2.	$a < b \ \vee \ a = b \ \vee \ b < a$	(trichotomy)
3.	$a < b \Rightarrow a + c < b + c$	(sum)
4.	$a < b \ \& \ c > 0 \Rightarrow ac < bc$	(product law)
5.	$\forall x : \mathbb{R}^*. \ \forall n : \mathbb{N}^+. \ \exists b : \mathbb{R}^*. \ b > 0 \ \& \ b^n = a$	(root axiom)

Definition An element of $\mathbb{R}$ is a *positive infinitesimal iff* is it less than every positive real and greater than 0. It is *negative iff* it is greater than every negative real. The real number 0 is also considered to be an infinitesimal, the *zero infinitesimal*.

Two elements of $\mathbb{R}^*$ are infinitely close, $a \approx b$, *iff* their difference, $a - b$, is infinitesimal. We write $a \not\approx b$ when they are not infinitely close.

Standard Part Axiom: Every hyperreal number is infinitely close to exactly one real number.

Standard Part Operator: We denote the *standard part* of a hyperreal a by $\mathrm{st}(a)$.

Facts: 1) $\mathrm{st}(a) \in \mathbb{R}$, $b = \mathrm{st}(b) + \epsilon$ for some infinitesimal ϵ if b is a finite hyperreal number.

2) If $a \in \mathbb{R}$, then $\mathrm{st}(a) = a$.

The Function Axiom $\forall f : \mathbb{R} \to \mathbb{R}. \ \exists f^* : \mathbb{R}^* \to \mathbb{R}^*$ called the *natural extension* of f. We discuss this extension when we examine examples. The idea is that for $x : \mathbb{R}$, $f(x) = f^*(x)$.

Solution Axiom The *hyperreal graph* of $y = f(x)$ is the set of hyperreal numbers (x_0, y_0) such that $y_0 = f^*(x_0)$.

References

[1] Robert L. Constable. Virtual evidence: A constructive semantics for classical logics. Technical Report arXiv:1409.0266, Computing and Information Science Technical Reports, Cornell University, 2014.

[2] K. Gödel. Über eine noch nicht benütste erweiterung des finiten standpunktes. *Dialectica*, 12:280–287, 1958.

[3] S. C. Kleene. *Introduction to Metamathematics*. D. Van Nostrand, Princeton, 1952.