CS6180 Lecture 20b

Robert Constable

October 31, 2017

1 Topics

(1) Challenge - Can there be a countable model of the real numbers? Explain.

(2) Axioms for the real numbers $\mathbb{R}$.

(3) Axioms for the hyperreal numbers $\mathbb{R}^*$.

(4) Defining infinitesimals.

2 Axioms for the Real Numbers, $\mathbb{R}$

$\mathbb{R}$ is the set of all reals, $\mathbb{P}$ the positive reals.

A1. $\forall x, y : \mathbb{R}.\ (x + y = y + x)$
A2. $\forall x, y, z : \mathbb{R}.\ (x + y) + z = x + (y + z)$
A3. $0 \in \mathbb{R}\ \&\ \forall x. : \mathbb{R}.\ (x + 0 = x)$
A4. $\forall x : \mathbb{R}.\ \exists y : \mathbb{R}.\ (x + y = 0)$
A5. $\forall x, y : \mathbb{R}.\ (x \cdot y = y \cdot x)$
A6. $\forall x, y, z : \mathbb{R}.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
A7. $1 \in \mathbb{R}.\ \&\ 1 \neq 0\ \&\ \forall x : \mathbb{R}.\ (x \cdot 1 = x)$
A8. $\forall x : \mathbb{R}.\ (x \neq 0 \Rightarrow \exists y : \mathbb{R}.\ (x \cdot y = 1)$
A9. $\forall x, y, z : \mathbb{R}.\ (x \cdot (y + z) = x \cdot y + x \cdot z)$

Axioms of Order (ordered field)

$\forall x, y : \mathbb{P}.(x + y) \in \mathbb{P} \quad \& \quad (x \cdot y) \in \mathbb{P} \quad \& \quad -x \notin \mathbb{P}$

$\forall x : \mathbb{R}.\ x = 0 \ \vee \ x \in \mathbb{P} \ \vee \ -x \in \mathbb{P}$

Completeness Axiom - let S be a set of reals.

$\forall S.\ (S \subseteq \mathbb{R} \Rightarrow \exists y : \mathbb{R}.\ y$ *is the least upper bound of* S*)*

We denote y by $\sup(S)$.

Archimedes Principle

$\forall x : \mathbb{R}.\ \exists n : \mathbb{N}.\ (x < n)$

Challenge: Find another more compact axiomatization of $\mathbb{R}$.

3 Notes on Infinitesimals and Hyperreal Numbers, $\mathbb{R}^*$

$0 < x < \frac{1}{n}$	$n > 0$	x is a positive infinitesimal
$\frac{-1}{n} < x < 0$	$n > 0$	x is a negative infinitesimal

I. Axioms for Infinitesimals

(a) There is a positive infinitesimal.

(b) If ϵ is a positive infinitesimal then $-\epsilon$ is a negative infinitesmial, and $r + \epsilon$ is hyperreal but not real.

(c) If $\epsilon > 0$ and $a > 0$, then $a \cdot \epsilon$ is a positive infinitesimal.

(d) If $\epsilon > 0$ then $\frac{1}{\epsilon}$ is a positive infinitesimal and $\frac{-1}{\epsilon}$ is a negative infinitesimal.

II. Algebraic Axioms for Hyperreals, $\mathbb{R}^*$

(a) $\forall x : \mathbb{R}.\ x \in \mathbb{R}^*$

(b) $\forall x, y : \mathbb{R}^*.\ x + y,\ x \cdot y,\ x - y$ belong to $\mathbb{R}^*$

(c) $\forall x : \mathbb{R}^*.\ x \neq 0 \Rightarrow \frac{1}{x} \in \mathbb{R}^*$

(d) The associative, commutative, distributive, and identity laws for $\mathbb{R}$ hold for $\mathbb{R}^*$.

III. Order Axioms for $\mathbb{R}^*$

1.	$a < b \ \& \ b < c \Rightarrow a < c$	(transitivity)
2.	$a < b \ \vee \ a = b \ \vee \ b < a$	(trichotomy)
3.	$a < b \Rightarrow a + c < b + c$	(sum)
4.	$a < b \ \& \ c > 0 \Rightarrow ac < bc$	(product law)
5.	$\forall x : \mathbb{R}^*. \ \forall n : \mathbb{N}^+. \ \exists b : \mathbb{R}^*. \ b > 0 \ \& \ b^n = a$	(root axiom)

Definition An element of $\mathbb{R}$ is a *positive infinitesimal iff* is it less than every positive real and greater than 0. It is *negative iff* it is greater than every negative real. The real number 0 is also considered to be an infinitesimal, the *zero infinitesimal*.

Two elements of $\mathbb{R}^*$ are infinitely close, $a \approx b$, *iff* their difference, $a - b$, is infinitesimal. We write $a \not\approx b$ when they are not infinitely close.

Standard Part Axiom: Every hyperreal number is infinitely close to exactly one real number.

Standard Part Operator: We denote the *standard part* of a hyperreal a by $\mathrm{st}(a)$.

Facts:

1) $\mathrm{st}(a) \in \mathbb{R}$, $b = \mathrm{st}(b) + \epsilon$ for some infinitesimal ϵ if b is a finite hyperreal number.

2) If $a \in \mathbb{R}$, then $\mathrm{st}(a) = a$.

The Function Axiom $\forall f : \mathbb{R} \to \mathbb{R}. \ \exists f^* : \mathbb{R}^* \to \mathbb{R}^*$ called the *natural extension* of f. We discuss this extension when we examine examples. The idea is that for $x : \mathbb{R}$, $f(x) = f^*(x)$.

Solution Axiom The *hyperreal graph* of $y = f(x)$ is the set of hyperreal numbers (x_0, y_0) such that $y_0 = f^*(x_0)$.