Lecture 20a: Constructive Nonstandard Models

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Abstract

Warning: Some of the ideas suggested here are experimental. We have not implemented them in type theory to access their value. We would prefer to have an account that can preserve the computational content of first-order statements when possible and reveal the obstacles to a constructive interpretation when necessary.

We will discuss the classical theory of nonstandard models of first-order theories. This is the foundation for the *nonstandard analysis* that we discussed in Lecture 19. The core of this investigation is an account of classical first-order logic. This requires us to have a constructive understanding of the law of excluded middle which can be given using the concept of *virtual evidence*.

We will also briefly discuss how to include the notion of a *constructive infinitesimal* which will allow us to give a simple definition of continuity for computable functions on the constructive reals.

Key Words: constructive reals, model theory, nonstandard models, infinitesimals.

1 Introduction

We can trace interest in nonstandard models back to results of Cantor as he was investigating set theory. One of his early results is the claim that there are the same number of points in the line segment ab where a and b are known to be separated as there are in the unit square with side ab. Another early result was Skolem's theorem that if a first order theory has a model, then it has an uncountable model.

The calculus was developed by Leibniz and Newton using the notion of "infinitely small numbers" called *infinitesimals* nowadays. In the rigorous development of the calculus, we typically use the "epsilon delta" arguments rather than infinitesimals. Here we repeat the comparison between a standard approach and the nonstandard approach to the notion of a *continuous function* on the interval $[a, b]$.

Definition 1: A function from reals to reals is *continuous* at x if and only if for every ϵ greater than 0, there is a real δ greater than 0, depending on ϵ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. We say that $lim f(x) = f(c)$ *as x goes to c.*

Definition 2: A function from reals to reals is continuous at a point c if and only if whenever x is *infinitely close* to c, $f(x)$ is infinitely close to $f(c)$.

The second definition sounds very intuitive and is simpler, but what does the phrase "infinitely close" mean? If we have the idea of an infinitesimal around, as Leibniz did, then we know – the two values differ only by an infinitesimal.

Classical theorems in *model theory* provide a precise definition of the concept of an infinitesimal real number. This subject relies heavily on nonconstructive results about classical first-order logic. So the first order of business for us is to understand classical first-order logic using the constructive one. We will take up this topic first.

We also need to compare classical set theory and type theory. It is very interesting that Aczel has given a constructive account of set theory in type theory in a series of accessible articles [1, 2, 3, 4]. But we will not study this topic.

2 Classical First-Order Logic defined in Intuitionistic FOL

The only nonconstructive axiom in classical first-order logic is the *law of excluded middle*, $P \lor \neg P$. We treat this rule using virtual evidence, writing the rule as $\forall P : Prop.\{P \lor \neg P\}$.

Below are the axioms from Kleene's book with axiom 8 modified to use virtual evidence. This means that the only evidence is the element $\star$, which represents "squashing" the constructive evidence down to a single token. They are listed here to illustrate formal logic.

2.1 Heyting Arithmetic Axioms from Kleene

Propositional Calculus Axioms

1a. $A \Rightarrow (B \Rightarrow A)$.
1b. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$.
Inference Rule 2:
$$\frac{A, A \Rightarrow B}{B}$$
3. $A \Rightarrow (B \Rightarrow A\&B)$.
4a. $(A\&B) \Rightarrow A$.
4b. $(A\&B) \Rightarrow B$.
5a. $A \Rightarrow (A \vee B)$.
5b. $B \Rightarrow (A \vee B)$.
6. $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$.
7. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \sim B) \Rightarrow \sim A)$.
8. $\sim\sim A \Rightarrow \{A\}$. **classical**
Note that $\{\forall A : Prop.(A \ \vee \ \sim A)\}$ is not constructively true.

Predicate Calculus Axioms

Inference Rule9:
$$\frac{C \Rightarrow A(x)}{C \Rightarrow \forall x.A(x)}$$
10. $\forall x.A(x) \Rightarrow A(t)$.
11. $A(t) \Rightarrow \exists x.A(x)$.
Inference Rule 12:
$$\frac{A(x) \Rightarrow C}{\exists x.A(x) \Rightarrow C}$$

Number Theory Axioms

13. $A(0)\&\forall x.(A(x) \Rightarrow A(x')) \Rightarrow A(x)$.
14. $a' = b' \Rightarrow a = b$.
15. $\sim (a' = 0)$.
16. $a = b \Rightarrow (a = c) \Rightarrow b = c$.
17. $a = b \Rightarrow a' = b'$.
18. $a + 0 = a$.
19. $a + b' = (a + b)'$.
20. $a \times 0 = 0$.
21. $a \times b' = (a \times b) + a$.

How do we use these axioms to create proofs? Kleene gives this example on page 85. This is an example of what logicians call a Hilbert style axiom system in which there is only one proof rule in addition to the axioms. The is the rule of *modus ponens*, if we know A and $A \Rightarrow B$, then we can deduce B.

1.$A \Rightarrow (A \Rightarrow A)$. Axiom schema 1a.
2.$(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A)$. Schema 1b.
3.$(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$. Axiom schema 1a.
4.$(A \Rightarrow A)$. Rules 2,4,3.

References

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