Lecture 18: The Constructive Real Numbers Continued

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Abstract

This lecture continues our discussion of the constructive reals and intuitionistic reals. We look briefly at other options beyond the approach taken in the book *Constructive Analysis* by Bishop and Bridges which is a resource for this course available on the PRL project page (www.nuprl.org). Another resource is the *calculator* for the constructive reals provided in Nuprl as a public resource. It uses algorithms verified in Nuprl by Dr. Bickford. This calculator relies on an implementation of bignums and relies on the Nuprl software stack which includes Lisp at the lowest level. We have high confidence in this resource which we have been using for over a decade. It is likely that the very first verified implementation of the reals numbers was done in 1985 and reported on in Chapter 11 of the book *Implementing Mathematics* from 1986.

In the next lecture we will look at nonstandard models of the reals which include elements called infinitessimals. We will consider them in a constructive setting as well.

1 Introduction

We have made the case that the constructive real numbers are useful creating reliable *cyber physical systems*, and that they are a fundamental concept in computational mathematics. We let $\mathbb{R}$ denote the constructive reals. Achieving a robust and applicable implementation has been a topic of interest in computer science at least since 1991 when the Nuprl system provided the first implementation of them [6]. Since that time the development and application of constructive analysis has been a regular topic of research using the Cornell type theory.

We have also had a brief look at the new result about the connectedness of the reals discovered by Mark Bickford and verified using Nuprl. This theorem is a fundamental result in constructive analysis that as far as we know is original to Dr. Bickford. It is a result that Brouwer would have been very pleased to discover.

2 Constructive and Intuitionistic Real Numbers

As we have noted before, a foundational understanding of the real numbers is a topic of broad interest in mathematics, logic, and computer science. Perhaps the most accessible approach to the classical reals as well is to think about how we might compute with them. We are familiar with the decimal expansions of several important reals, such as π which begins with these digits:

3.1415926535897932384626433832795028841971693993751058209749445923078164...

It is possible to compute as many digits of *pi* as anyone would need to use or want to see. So far at least 12.1 trillion digits of *pi* have been computed. We can easily find ten thousand digits of e on the web, it looks like this:

2.7182818284590452353602874713526624977572470936999595749669676277240766...

It would not be so much fun to multiply these two approximations together to approximate $\pi \times e$, but we know how to do it. One intuitive grasp of the constructive reals might be in terms of these *never ending decimal approximations.* This was an approach that Turing investigated [9]. However as Turing learned, this is not a good definition of the computable reals if we want to prove properties of them and build a theory of computable real numbers to support the calculus. From implementing results of Brouwer in Nuprl, we know that if we use algorithms to define the sequence of digits, say Turing Machines, we do not arrive at an adequate theory rich enough to implement all of the important and practical concepts of Brouwer's intuitionistically computable real numbers nor rich enough to validate all of Brouwer's discoveries about computing with these numbers.

Attempting to understand the Euclidean plane in terms of points and lines leads to insights about the constructive reals, but it does not easily lead to the fundamental concepts discovered by Brouwer. Relating geometric points to constructive reals has been a source of productive insights about the nature of the real numbers, but the account we have thus far is not fully satisfactory and leaves open some questions that are at least a thousand years old. On the other hand, our implementation of the constructive real numbers is an important tool in our computational investigation of geometry.[1]

There are several paths from geometry to the constructive real numbers. One path originates with Newton and Leibniz, motivated in part by the need to compute. The path to formulating the right axioms on the reals usually starts with the algebraic operations such as adding, subtracting, multiplying, and dividing real numbers. These axioms state that the real numbers have the algebraic properties of a field. The rational numbers also form a field, so this path also requires a property that distinguishes the reals from the rational numbers. One such property is captured in the *completeness axiom*, i.e. that every bounded non-empty subsets of the reals has a least upper bound.

There are other approaches to understanding the real numbers that are more geometric and topological. It is appealing to to formulate these properties in type theory and discover their "computational meaning" in various ways. This is not a closed topic even though some of the concepts involved have been studied for hundreds of years. Relatively modern mathematicians such as Weyl [12] and Brouwer [11, 10, 5] and Poincaré [11] have written extremely influential studies of the reals. We were very influenced by the pragmatic approach of the American analyst Errett Bishop [3, 2] and his close collaborator Douglas Bridges [4]. However, in 2016 we changed the Nuprl account to match L.E.J. Brouwer's definition of the reals, leading us to a fully intuitionistic account. We did this on pragmatic grounds, the theory is more useful because it provides better results about continuity. We also took this step because it enriches our type theory in ways that have considerable computational value. We reported on these results in LICS 2017 [1].

If there is time in the course, we will explore other paths to the constructive real numbers that reveal other important aspects of this concept and other paths that lead to them. The path from Euclidean geometry traces the actual history of this very important concept. This data type is so important in science generally that we have made available on the Nuprl web page, as a public service, a calculator that is provably correct and allows researchers who need to know the exact real number values in critical computations to access our verified implementation.

[1]We have already noted that such investigations are very subtle and we can be led astray, thinking the Euclid Proposition 2 is not constructively true, when we know that it really is.

3 Bishop's Constructive Reals

In 1967 Errett Bishop published his landmark book *Foundations of Constructive Analysis* [2]. One of our goals in building the Nuprl proof assistant was to formalize the basic results of this book, and in Chapter 11 of our 1986 book, Implementing Mathematics [7], we started on this task with more results to follow in the 1992 article by Chirimar and Howe, "Implementing Constructive Real Analysis" [6]. We found that formalizing the basic concepts from Bishop's book was tractable in Nuprl. Bishop came to visit us and provided additional motivation to pursue this investigation. His basic approach did not involve working with infinite precision decimal reals but rather following Brouwer and Heyting by using converging sequences of rational numbers. It is somewhat delicate to translate this representation into unbounded decimal notations for reasons we will briefly discuss.

Bishop's approach was extended by Douglas Bridges to cover more of real analysis, and that led to a new book by Bishop and Bridges entitled simply *Constructive Analysis*, [4]. Springer-Verlag gave the PRL group permission to put Chapter 2 of this book on the PRL web page (www.nuprl.org) where you can now read it. This lecture is based on Chapter 2 of that book. Readers will see that we do not use infinite precision decimals to define the real numbers. The article by John Myhill, "What is a Real Number?" [8] discusses the hazards of using decimal representations of the reals. It is available in the readings provided with this lecture. This lecture will stress the advantages of the approach taken by Bishop and Bridges. We will also remark on safe ways to convert Bishop's representation to infinite precision decimal representations, although there are Bishop reals for which we cannot find a decimal representation. These are convenient for conveying real numbers in a style similar to floating point reals.

It is interesting that Turing provided an account of the constructive reals using decimal representations. One of my most famous PhD students, Turing award winner Edmund Clarke, gives a lecture on Turing's reals that you can find on-line using Google.

References

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