

CHAPTER II.

THE THEORY OF LOGICAL TYPES.

THE theory of logical types, to be explained in the present Chapter, recommended itself to us in the first instance by its ability to solve certain contradictions, of which the one best known to mathematicians is Burali-Forti's concerning the greatest ordinal. But the theory in question is not wholly dependent upon this indirect recommendation: it has also a certain consonance with common sense which makes it inherently credible. In what follows, we shall therefore first set forth the theory on its own account, and then apply it to the solution of the contradictions.

I. *The Vicious-Circle Principle.*

An analysis of the paradoxes to be avoided shows that they all result from a certain kind of vicious circle*. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. Thus, for example, the collection of *propositions* will be supposed to contain a proposition stating that "all propositions are either true or false." It would seem, however, that such a statement could not be legitimate unless "all propositions" referred to some already definite collection, which it cannot do if new propositions are created by statements about "all propositions." We shall, therefore, have to say that statements about "all propositions" are meaningless. More generally, given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. By saying that a set has "no total," we mean, primarily, that no significant statement can be made about "all its members." Propositions, as the above illustration shows, must be a set having no total. The same is true, as we shall shortly see, of propositional functions, even when these are restricted to such as can significantly have as argument a given object *a*. In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting.

* See the last section of the present Chapter. Cf. also H. Poincaré, "Les mathématiques et la logique," *Revue de Métaphysique et de Morale*, Mai 1906, p. 307.

The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. Arguments which are condemned by the vicious-circle principle will be called "vicious-circle fallacies." Such arguments, in certain circumstances, may lead to contradictions, but it often happens that the conclusions to which they lead are in fact true, though the arguments are fallacious. Take, for example, the law of excluded middle, in the form "all propositions are true or false." If from this law we argue that, because the law of excluded middle is a proposition, therefore the law of excluded middle is true or false, we incur a vicious-circle fallacy. "All propositions" must be in some way limited before it becomes a legitimate totality, and any limitation which makes it legitimate must make any statement about the totality fall outside the totality. Similarly, the imaginary sceptic, who asserts that he knows nothing, and is refuted by being asked if he knows that he knows nothing, has asserted nonsense, and has been fallaciously refuted by an argument which involves a vicious-circle fallacy. In order that the sceptic's assertion may become significant, it is necessary to place some limitation upon the things of which he is asserting his ignorance, because the things of which it is possible to be ignorant form an illegitimate totality. But as soon as a suitable limitation has been placed by him upon the collection of propositions of which he is asserting his ignorance, the proposition that he is ignorant of every member of this collection must not itself be one of the collection. Hence any significant scepticism is not open to the above form of refutation.

The paradoxes of symbolic logic concern various sorts of objects: propositions, classes, cardinal and ordinal numbers, etc. All these sorts of objects, as we shall show, represent illegitimate totalities, and are therefore capable of giving rise to vicious-circle fallacies. But by means of the theory (to be explained in Chapter III) which reduces statements that are verbally concerned with classes and relations to statements that are concerned with propositional functions, the paradoxes are reduced to such as are concerned with propositions and propositional functions. The paradoxes that concern propositions are only indirectly relevant to mathematics, while those that more nearly concern the mathematician are all concerned with *propositional functions*. We shall therefore proceed at once to the consideration of propositional functions.

II. *The Nature of Propositional Functions.*

By a "propositional function" we mean something which contains a variable x , and expresses a *proposition* as soon as a value is assigned to x . That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. It agrees with the ordinary functions of mathematics in the fact of containing an unassigned variable: where it differs is in the fact that the values of the function are propositions. Thus *e.g.* " x is a man" or " $\sin x = 1$ " is a propositional function. We shall find that it is possible to incur a vicious-circle fallacy at the very outset, by admitting as possible arguments to a propositional function terms which presuppose the function. This form of the fallacy is very instructive, and its avoidance leads, as we shall see, to the hierarchy of types.

The question as to the nature of a function* is by no means an easy one. It would seem, however, that the essential characteristic of a function is *ambiguity*. Take, for example, the law of identity in the form " A is A ," which is the form in which it is usually enunciated. It is plain that, regarded psychologically, we have here a single judgment. But what are we to say of the object of the judgment? We are not judging that Socrates is Socrates, nor that Plato is Plato, nor any other of the definite judgments that are instances of the law of identity. Yet each of these judgments is, in a sense, within the scope of our judgment. We are in fact judging an ambiguous instance of the propositional function " A is A ." We appear to have a single thought which does not have a definite object, but has as its object an undetermined one of the values of the function " A is A ." It is this kind of ambiguity that constitutes the essence of a function. When we speak of " ϕx ," where x is not specified, we mean one value of the function, but not a definite one. We may express this by saying that " ϕx " *ambiguously denotes* ϕa , ϕb , ϕc , etc., where ϕa , ϕb , ϕc , etc., are the various values of " ϕx ."

When we say that " ϕx " ambiguously denotes ϕa , ϕb , ϕc , etc., we mean that " ϕx " means one of the objects ϕa , ϕb , ϕc , etc., though not a definite one, but an undetermined one. It follows that " ϕx " only has a well-defined meaning (well-defined, that is to say, except in so far as it is of its essence to be ambiguous) if the objects ϕa , ϕb , ϕc , etc., are well-defined. That is to say, a function is not a well-defined function unless all its values are already well-defined. It follows from this that no function can have among its values anything which presupposes the function, for if it had, we could not regard the objects ambiguously denoted by the function as definite until the function was definite, while conversely, as we have just seen, the function cannot be

* When the word "function" is used in the sequel, "propositional function" is always meant. Other functions will not be in question in the present Chapter.