Tensors and Exponentials

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Definition (Left Exponential)**.** Given two objects A and B of a multicategory, an object E along with a multimorphism $e : [A, E] \to B$ is a left exponential of A and B if for every multimorphism $f : [A] +\!\!+ \vec{C} \to B$ there exists a unique multimorphism $f' : \vec{C} \to E$ with the property that $\triangle_{id_A, f'}\, e$ equals f.

Remark. Left exponentials are essentially unique, and as such E is often denoted as $A \multimap B$ and e as $\mathsf{eval}_{A,B} : [A, A \multimap B] \to B$. The uniquely induced multimorphism for f is often denoted λf.

Definition (Right Exponential)**.** Given two objects A and B of a multicategory, an object E along with a multimorphism $e : [E, A] \to B$ is a right exponential of A and B if for every multimorphism $f : \vec{C} +\!\!+ [A] \to B$ there exists a unique multimorphism $f' : \vec{C} \to E$ with the property that $\triangle_{f', id_A}\, e$ equals f.

Remark. Right exponentials are essentially unique, and as such E is often denoted as $B \multimapinv A$ and e sometimes as $\mathsf{lave}_{A,B} : [B \multimapinv A, A] \to B$. The uniquely induced multimorphism for f is sometimes denoted $\reflectbox{$\lambda$} f$.

Definition (Left-Closed, Right-Closed, and Biclosed)**.** A multicategory $\mathbf{C}$ is left-closed if every pair of objects has a left exponential. A multicategory $\mathbf{C}$ is right-closed if every pair of objects has a right exponential. A multicategory $\mathbf{C}$ is biclosed if it is both left-closed and right-closed. In this case, $A \multimap (C \multimapinv B)$ is canonically isomorphic to $(A \multimap C) \multimapinv B$, and so the notation $A \multimap C \multimapinv B$ is often used.

Definition (Tensor)**.** Given a list of objects $\vec{B}$ of a multicategory, an object T along with a morphism $t : \vec{B} \to T$ is a tensor of $\vec{B}$ if for every multimorphism $f : \vec{A} +\!\!+ \vec{B} +\!\!+ \vec{C} \to D$ there exists a unique morphism $f' : \vec{A} +\!\!+ [T] +\!\!+ \vec{C} \to D$ with the property that $\triangle_{\overrightarrow{id_A}, t, \overrightarrow{id_C}}\, f'$ equals f.

Remark. Tensors are essentially unique. As such, when $\vec{B}$ is empty, T is often denoted as I. When $\vec{B}$ is a singleton list $[B]$, T is necessarily isomorphic to B. When $\vec{B}$ contains more than one object $B_1, \ldots, B_n$, T is often denoted as $B_1 \otimes \cdots \otimes B_n$. In all cases, t is sometimes denoted as $\mathsf{tuple}_{B_1,\ldots,B_n}$, and the multimorphism uniquely induced from f is sometimes denoted as $\mathsf{split}_{\vec{A};\vec{B};\vec{C}} f$.

One can also extend $\otimes$ into an operation on lists of unary multimorphisms $f_1 : [A_1] \to B_1, \ldots, f_n : [A_n] \to B_n$. The multimorphism denoted $f_1 \otimes \cdots \otimes f_n : [A_1 \otimes \cdots \otimes A_n] \to (B_1 \otimes \cdots \otimes B_n)$ is defined as

$$\mathsf{split}_{\varnothing;A_1,\ldots,A_n;\varnothing} \mathop{\triangle}_{f_1,\ldots,f_n} \mathsf{tuple}_{B_1,\ldots,B_n}$$

Definition (Representable)**.** A multicategory is representable if every list of objects has a tensor.

Definition (Strong Coproduct)**.** Given an I-indexed collection of objects $\{B_i\}_{i \in I}$, an object B along with an I-indexed collection of multimorphisms $\{k_i : [B_i] \to B\}$ is a strong coproduct of $\{B_i\}_{i \in I}$ if for every I-indexed collection of multimorphisms $\{f_i : \vec{A} +\!\!+ [B_i] +\!\!+ \vec{C} \to D\}$ there exists a unique morphism $f : \vec{A} +\!\!+ [B] +\!\!+ \vec{C} \to D$ with the property that $\triangle_{\overrightarrow{id_A}, k_i, \overrightarrow{id_C}}\, f$ equals f_i for all i in I.

Remark. Strong coproducts are essentially unique and are often denoted in the same manner as coproducts since every strong coproduct is necessarily a coproduct. Sometimes, though, the notation $\oplus$ is specifically used to distinguish strong coproducts from coproducts, in which case sometimes $\perp$ is used to denote the empty strong coproduct. Note that what makes a coproduct "strong" is the ability to tolerate additional left and right contexts $\vec{A}$ and $\vec{C}$. More generally, strong colimits are colimits with this additional ability.

Remark. Products need no such similar "strong" variant. Sometimes, though, products in multicategories are denoted with $\parr$ and $\top$ rather than $\times$ and 1, with the latter reserved for only when products and tensors coincide.

Definition (Representable Cartesian Multicategory)**.** A representable cartesian multicategory is a representable multicategory in which an object is a tensor object of a list of objects if and only if it is a product object of that list of objects.

Definition (Skewed Operad)**.** A skewed operad is an operad $\mathbf{P}$ with the following additional components:

Skew Codomain For every multimorphism p, a natural number $c(p) \in \mathbb{N}$ such that

$$c(id) = 1 \qquad \text{and} \qquad c\left(\mathop{\triangle}_{p_1,\ldots,p_n} p'\right) = \sum_{i \in \{1,\ldots,c(p')\}} c(p_{\sigma_{p'}(i)})$$

Skew Identity For every natural number n, an n-ary multimorphism sid_n such that $c(sid_n)$ equals n

Skew Composition For every n-ary multimorphism p and $c(p)$-ary multimorphism p', an n-ary multimorphism $p\,;p'$ such that $c(p\,;p')$ equals $c(p')$

Skew-Identity Interchange id equals sid_1, and $\triangle_{sid_{m_1},\ldots,sid_{m_n}} sid_n$ equals $sid_{m_1+\cdots+m_n}$

Skew-Composition Interchange $\triangle_{p_1\,;p'_1,\ldots,p_n\,;p'_n} p_0\,;p'_0$ equals $\left(\triangle_{p_1,\ldots,p_n} p_0\right); \left(\triangle_{p'_{\sigma_{p_0}(1)},\ldots,p'_{\sigma_{p_0}(c(p_0))}} p'_0\right)$

Example. Cart is the skewed operad whose components are as follows:

Multimorphisms A multimorphism is a triple $\langle n \in \mathbb{N}, m \in \mathbb{N}, \sigma : \{1,\ldots,m\} \to \{1,\ldots,n\}\rangle$

Arity The arity of $\langle n, m, \sigma\rangle$ is n

Identity id is $\langle 1, 1, \lambda i \in \{1\}.\, i\rangle$

Composition $\triangle_{\langle n_1,m_1,\sigma_1\rangle,\ldots,\langle n_n,m_n,\sigma_n\rangle}\langle n, m, \sigma\rangle$ is $\left\langle \sum_{i\in\{1,\ldots,n\}} n_i, \sum_{i\in\{1,\ldots,m\}} m_{\sigma(i)}, \triangle_{\sigma_1,\ldots,\sigma_n} \sigma\right\rangle$ where

$$\left(\triangle_{\sigma_1,\ldots,\sigma_n} \sigma\right)\left(i \in \left\{1,\ldots,\textstyle\sum_{i\in\{1,\ldots,m\}} m_{\sigma(i)}\right\}\right) = \sigma_{\sigma(j)}(k) + \textstyle\sum_{i\in\{1,\ldots,\sigma(j)-1\}} n_i$$
$$\text{where} \quad j \in \{1,\ldots,m\}, \quad k \in \{1,\ldots,m_{\sigma(j)}\}, \quad i = k + \textstyle\sum_{i\in\{1,\ldots,j-1\}} m_{\sigma(i)}$$

Skew Codomain $c(\langle n, m, \sigma\rangle)$ is m

Skew Identity sid_n is $\langle n, n, \lambda i \in \{1,\ldots,n\}.\, i\rangle$

Skew Composition $\langle n, m, \sigma\rangle\,;\langle m, k, \sigma'\rangle$ is $\langle n, k, \lambda i \in \{1,\ldots,k\}.\, \sigma(\sigma'(i))\rangle$

Definition (Skewed Multifunctor)**.** A skewed multifunctor between skewed operad is a multifunctor that preserves skew codomains, skew identities, and skew compositions.

Example. Sym is the sub-skewed-operad of **Cart** comprised of the morphisms $\langle n, m, \sigma\rangle$ in which m equals n and σ is a bijection.

Definition (Reindexable Multicategory)**.** Given a skewed operad $\mathbf{P}$ with a skewed multifunctor $\sigma : \mathbf{P} \to \mathbf{Cart}$, a σ-reindexable multicategory is a multicategory $\mathbf{C}$ along with, for every n-ary multimorphism p of $\mathbf{P}$ and every list of objects $[A_1,\ldots,A_n]$ of $\mathbf{C}$, an assignment of each multimorphism $f : [A_{\sigma_p(1)},\ldots,A_{\sigma_p(c(p))}] \to B$ of $\mathbf{C}$ to a "reindexed" multimorphism $\theta_p(f) : [A_1,\ldots,A_n] \to B$ of $\mathbf{C}$ such that the following properties hold:

$$\forall \ldots.\;\; f = \theta_{(vid_n)}(f) \qquad\qquad \forall \ldots.\;\; \theta_p(\theta_{p'}(f)) = \theta_{(p\,;p')}(f)$$

$$\forall \ldots. \quad \mathop{\triangle}_{\theta_{p_1}(f_1),\ldots,\theta_{p_n}(f_n)} \theta_{p'}(f') = \theta_{(\triangle_{p_1,\ldots,p_n} p')}\left(\mathop{\triangle}_{f_{\sigma_{p'}(1)},\ldots,f_{\sigma_{p'}(c(p'))}} f'\right)$$

Definition (Cartesian Multicategory)**.** A cartesian multicategory is a multicategory reindexable along the identity skewed multifunctor on **Cart**.

Definition (Symmetric Multicategory)**.** A symmetric multicategory is a multicategory reindexable along the inclusion skewed multifunctor from **Sym** to **Cart**.

Remark. Note that these new definitions provide another definition for representable cartesian multicategory. In fact, this new definition is equivalent to the one given before. In particular, in a cartesian multicategory, all tensors are necessary products, and in a representable multicategory, if all tensors are products then the multicategory can be given a cartesian structure, and these two constructions are inverses of each other.