Practice 8

Ross Tate

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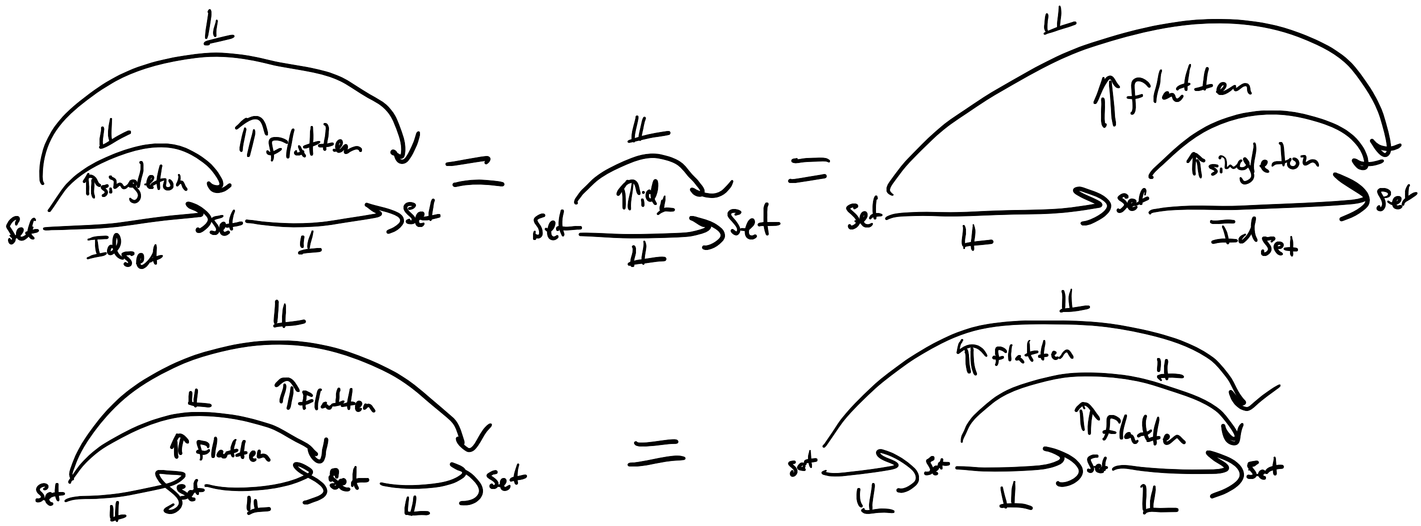
Exercise 1. `singleton` : **Set** $\Rightarrow \mathbb{L}$ and `flatten` : $\mathbb{L}\mathbb{L} \Rightarrow \mathbb{L}$ are two particularly important natural transformations pertaining to lists. The following equate various compositions of these natural transformations: Write each of the



diagrams as polymorphic programs of the form λ`l` $: \tau. \ldots$ for some type τ (referencing a type variable α) using the following "library" functions *with explicit subscripts* (i.e. use explicit type arguments):

$\texttt{map}_{\alpha,\beta} : (\alpha \to \beta) \to (\mathbb{L}\alpha \to \mathbb{L}\beta)$

$\texttt{singleton}_{\alpha} : \alpha \to \mathbb{L}\alpha$

$\texttt{flatten}_{\alpha} : \mathbb{L}\mathbb{L}\alpha \to \mathbb{L}\alpha$

Then prove the above equalities.

Exercise 2. Let $\mathbf{Map}(\mathbf{C}, \mathbf{D})$ be the following *lax relational* category:

- an object V maps each object A of $\mathbf{C}$ to an object $V(A)$ of $\mathbf{D}$
- a morphism E from V_1 to V_2 maps each morphism $m : A \to B$ of $\mathbf{C}$ to a morphism $E(m) : V_1(A) \to V_2(B)$
- a (possibly empty) path $V_0 \xrightarrow{E_1} V_1 \xrightarrow{\cdots} V_{n-1} \xrightarrow{E_n} V_n$ composes to $V_0 \xrightarrow{E} V_n$ whenever

$$\forall A_0 \xrightarrow{m_1} A_1 \xrightarrow{\cdots} A_{n-1} \xrightarrow{m_n} A_n.\ E_1(m_1) \,;\ldots;\, E_n(m_n) = E(m_1 \,;\ldots;\, m_n)$$

This is a lax relational category because composition is defined as a relation, rather than function, from paths to morphisms (that satisfies laws akin to a lax notion of associativity/identity).

Prove that functors from $\mathbf{C}$ to $\mathbf{D}$ precisely coincide with endomorphisms $E : V \to V$ of $\mathbf{Map}(\mathbf{C}, \mathbf{D})$ with the property that all paths of the form $E^* : V \to V$, i.e. arbitrary repetitions of E, compose to E. Prove that natural transformations from the functor coinciding with $V_1 \xrightarrow{E_1} V_1$ to the functor coinciding with $V_2 \xrightarrow{E_2} V_2$ precisely coincide with morphisms $T : V_1 \to V_2$ of $\mathbf{Map}(\mathbf{C}, \mathbf{D})$ with the property that all paths of the form $E_1^* T E_2^* : V_1 \to V_2$ compose to T.