Determined Relations

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May 6, 2018

Definition. Let Φ be a collection of "abstract symbols" each paired with a natural number n indicating the arity of the symbol. For example, $\Phi = \{\leq : 2\}$ has one abstract symbol, $\leq$, of arity 2, indicating that there should be a relation $\leq$ that is binary. Let Δ be a collection of "determinisms", each of which is an abstract symbol $R : n$ in Φ and a subset of $\{1, \ldots, n\}$. For example, $\Phi = \{+\!\sim : 3\}$ and $\Delta = \{+\!\sim : \{1, 2\}\}$ indicates that there should be a ternary relation $+=$ whose remaining arguments (i.e. its third argument) are uniquely determined by its first and second arguments together.

Given a Φ and Δ, the construct $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ is comprised of the following:

Object An object is a set A along with, for each symbol-arity pair $R : n \in \Phi$, a relation $R_A \subseteq A^n$ such that, for each determinism $R : J \in \Delta$, given any two n-tuples $\vec{a}$ and $\vec{a}'$ in R_A, if $\forall j \in J.\ a_j = a'_j$ then $\vec{a} = \vec{a}'$.

Morphism A morphism from $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$ to $\langle B, \{R_B\}_{R:n\in\Phi}\rangle$ is a function $f : A \to B$ such that, for every symbol-arity pair $R : n \in \Phi$, given any n-tuple $\vec{a}$ in R_A, the n-tuple $f(\vec{a})$ is in R_B (where $f(\vec{a})$ is shorthand for $\langle f(a_i)\rangle_{i\in\{1,\ldots,n\}}$).

Example. The construct $\mathbf{Rel}(\leq : 2, +\!\sim : 3)\mathbf{Det}(+\!\sim : \{1, 2\})$ is more explicitly comprised of the following:

Object An object is a set A along with a binary relation $\leq\ \subseteq A \times A$ and a ternary relation $+\!\sim\ \subseteq A \times A \times A$ such that whenever $a_1 + a_2 \sim a_3$ and $a'_1 + a'_2 \sim a'_3$ both hold, then if a_1 equals a'_1 and a_2 equals a'_2 then a_3 equals a'_3. In other words, whenever $a_1 + a_2 \sim a_3$ and $a_1 + a_2 \sim a'_3$ both hold, then a_3 equals a'_3.

Morphism A morphism from $\langle A, \leq, +\!\sim\rangle$ to $\langle B, \leq, +\!\sim\rangle$ is a function $f : A \to B$ such that, for all a_1 and a_2 in A, if $a_1 \leq a_2$ holds then $f(a_1) \leq f(a_2)$ holds, and for all a_1, a_2, and a_3 in A, if $a_1 + a_2 \sim a_3$ holds then $f(a_1) + f(a_2) \sim f(a_3)$ holds.

Remark. The category $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ is an epi-implicational subcategory of $\mathbf{Rel}(\Phi)$. In particular, there is one epi-implication for each determinism in Δ. For example, the epi-implication in $\mathbf{Rel}(+\!\sim : 3)$ for the determinism $+\!\sim : \{1, 2\}$ is the following epimorphism:

$$\langle\{x_1, x_2, x_3, x'_3\}, \{\langle x_1, x_2, x_3\rangle, \langle x_1, x_2, x'_3\rangle\}\rangle \xrightarrow{x'_3 \mapsto x_3} \langle\{x_1, x_2, x_3\}, \{\langle x_1, x_2, x_3\rangle\}\rangle$$

Consequently, $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ has an $(\mathrm{Epi}_I, \text{Initial Mono-Source}_I)$-factorization structure, meaning it has a factorization structure between its morphisms that are epic *in* $\mathbf{Rel}(\Phi)$ and its sources that are initial and monic *in* $\mathbf{Rel}(\Phi)$.

Note, however, that there are morphisms that are epic in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ but *not* epic in $\mathbf{Rel}(\Phi)$. One particularly important example is the following morphism in $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(\{1, 2\} \xmapsto{+\sim} \{3\})$:

$$\langle\{x_1, x_2\}, \varnothing\rangle \xrightarrow{\mathsf{total}} \langle\{x_1, x_2, x_3\}, \{\langle x_1, x_2, x_3\rangle\}\rangle$$

Given any morphism f from $\langle\{x_1, x_2, x_3\}, \{\langle x_1, x_2, x_3\rangle\}\rangle$ to some other object A in $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(+\!\sim : \{1, 2\})$, its mapping of x_3 is determined uniquely by its mapping of x_1 and x_2 because f must be relation-preserving and, in order for A to be contained in $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(+\!\sim : \{1, 2\})$, there can be at most one element a_3 of A such that $f(x_1) + f(x_2) \sim a_3$ holds. Thus total is epic in $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(+\!\sim : \{1, 2\})$. However, this same reasoning does not apply to $\mathbf{Rel}(+\!\sim : 3)$, and it is easy to construct a counterexample showing that total is not epic in $\mathbf{Rel}(+\!\sim : 3)$. This means that total belongs to Epi but *not* to Epi_I.

This fact is unfortunate because total encodes the implication that $+\!\sim$ must be total. That is, the implicational subcategory of $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(\{1, 2\} \xmapsto{+\sim} \{3\})$ satisfying total is the subcategory of $\mathbf{Rel}(+\!\sim : 3)$ comprised of the objects whose $+\!\sim$ relation is determined and total, i.e. specifies a function. This implicational subcategory is concretely isomorphic to $\mathbf{Magma}$, also known as $\mathbf{Alg}(2)$, so if we can show that total belongs to an $\mathcal{E}$ of some factorization structure on $\mathbf{Rel}(+\!\sim : 3)\mathbf{Det}(+\!\sim : \{1, 2\})$, then we will have a nicely-behaved unification of relations and algebras that simply views algebraic operators as total and determined relations.

Theorem. *If Δ is a set (rather than an arbitrary-sized collection), then the category* $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ *has an* $(\mathit{Epi}, \mathit{Extremal\ Mono\text{-}Source})$*-factorization structure.*

Proof. Unfortunately the collection Extremal Mono-Source for $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ is larger than Initial Mono-Source$_U$, so we cannot use the proof from the homework. While we could use factorization structures on *functors* (rather than categories), here we will prove it from first principles rather than introduce yet another concept.

For our first step, we classify (without proof) the epimorphisms of $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$. A morphism f from an object $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$ to an object $\langle B, \{R_B\}_{R:n\in\Phi}\rangle$ is epic in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ if and only if for every element b in B there is a proof that f $\mathsf{generates}^{\Delta}_{\{R_B\}_{R:n\in\Phi}}$ b built from the following inference rules:

$$\frac{a \in A}{f \ \mathsf{generates}^{\Delta}_{\{R_B\}_{R:n\in\Phi}} \ f(a)} \qquad \frac{\begin{array}{ccc} R : n \in \Phi & \langle b_1, \ldots, b_n\rangle \in R_B & k \in \{1, \ldots, n\} \\ R : J \in \Delta & \multicolumn{2}{c}{\text{for all } j \text{ in } J,\ f \ \mathsf{generates}^{\Delta}_{\{R_B\}_{R:n\in\Phi}} \ b_j} \end{array}}{f \ \mathsf{generates}^{\Delta}_{\{R_B\}_{R:n\in\Phi}} \ b_k}$$

For our second step, we classify (without proof) the extremal mono-sources of $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$. A source $\{f_i\}_{i\in I}$ from an object $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$ to objects $\langle B_i, \{R_i\}_{R:n\in\Phi}\rangle$ is an extremal mono-source in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ if and only if the following three properties hold:

Mono $\forall a, a' \in A.\ (\forall i \in I.\ f_i(a) = f_i(a')) \implies a = a'$

Initial $\forall R : n \in \Phi.\ \forall a_1, \ldots, a_n \in A.\ (\forall i \in I.\ \langle f_i(a_1), \ldots, f_i(a_n)\rangle \in R_i) \implies \langle a_1, \ldots, a_n\rangle \in R_A$

Extremal $\forall R : n \in \Phi.\ \forall R : J \in \Delta.\ \forall a \in J \to A.$
$(\forall i \in I.\ \exists \langle b_1, \ldots, b_n\rangle \in R_i.\ \forall j \in J.\ f_i(a_j) = b_j) \implies \exists \langle a'_1, \ldots, a'_n\rangle \in R_A.\ \forall j \in J.\ a_j = a'_j$

For our third step, we construct factorizations of sources. To do so, we first define the *set* of "expressions with free variables A from determinisms Δ of Φ", which we can only do if Δ itself is a set. Define $\mathsf{Expr}^{\Phi}_{\Delta}(A)$ to be the smallest set with the following disjoint injective functions (i.e. constructors):

$\mathsf{var} : A \hookrightarrow \mathsf{Expr}^{\Phi}_{\Delta}(A)$ for each $R : n$ in Φ, $R : J$ in Δ, and k in $\{1, \ldots n\}$, $\mathsf{op}^{R}_{k} : (J \to \mathrm{Expr}^{\Phi}_{\Delta}(A)) \hookrightarrow \mathsf{Expr}^{\Phi}_{\Delta}(A)$

Given a source $\{f_i\}_{i\in I}$ from an object $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$ to objects $\langle B_i, \{R_i\}_{R:n\in\Phi}\rangle$, for each i in I we inductively define the following $\overset{i}{\mapsto}$ relation between expressions $\mathsf{Expr}^{\Phi}_{\Delta}(A)$ and elements of B_i with the following inference rules:

$$\frac{a \in A}{\mathsf{var}(a) \overset{i}{\mapsto} f_i(a)} \qquad \frac{\begin{array}{ccc} R : n \in \Phi & \langle b_1, \ldots, b_n\rangle \in R_i & k \in \{1, \ldots, n\} \\ R : J \in \Delta & e : J \to \mathsf{Expr}^{\Phi}_{\Delta}(A) & \text{for all } j \text{ in } J,\ e_j \overset{i}{\mapsto} b_j \end{array}}{\mathsf{op}^{R}_{k}(e) \overset{i}{\mapsto} b_k}$$

Because every R_i satisfies the relevant determinisms in Δ, one can easily show that every $\overset{i}{\mapsto}$ relation is determined, meaning there is at most one $b \in B_i$ that a given expression maps to via $\overset{i}{\mapsto}$. However, the $\overset{i}{\mapsto}$ relations are not necessarily total, and so we define E to be $\{e \in \mathsf{Expr}^{\Phi}_{\Delta}(A) \mid \forall i \in I.\ \exists b \in B_i.\ e \overset{i}{\mapsto} b\}$. By definition, every $\overset{i}{\mapsto}$ relation is total on this subset, and since every $\overset{i}{\mapsto}$ relation is also determined, they each specify a function, say g_i, from E to B_i. Thus we have a source $\{g_i : E \to B_i\}_{i\in I}$ in $\mathbf{Set}$. Let $(e : E \to Q, \{m_i : Q \to B_i\}_{i\in I})$ be its (Epi, Mono-Source)-factorization. The set Q will be the underlying set of our factorization of $\{f_i\}_{i\in I}$ in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$. In order to define the relations on Q we observe that the construct $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ is *mono*topological, and so for each $R : n$ in Φ we define R_Q to be the relation $\{\langle q_1, \ldots, q_n\rangle \mid \forall i \in I.\ \langle m_i(q_1), \ldots, m_i(q_n)\rangle \in R_i\}$, which can easily be shown to satisfy the relevant determinisms in Δ because every R_i does. From the constructions of E, of $\{m_i\}_{i\in I}$, and of $\{R_Q\}_{R:n\in\Phi}$, one can easily show that the source $\{m_i : \langle Q, \{R_Q\}_{R:n\in\Phi}\rangle \to \langle B_i, \{R_i\}_{R:n\in\Phi}\rangle\}_{i\in I}$ is an extremal mono-source in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$. Lastly, it is easy to prove that every expression of the form $\mathsf{var}(a)$ belongs to the subset E, so there is a function from A to Q given by $a \mapsto e(\mathsf{var}(a))$. It is furthermore easy to prove that this function lifts to a morphism from $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$ to $\langle Q, \{R_Q\}_{R:n\in\Phi}\rangle$, and additionally that this morphism is epic in $\mathbf{Rel}(\Phi)\mathbf{Det}(\Delta)$ due to the constructions of E, of e, and of $\{R_Q\}_{R:n\in\Phi}$. And clearly this morphism has the property that, when composed with m_i for any i in I, results in f_i.

For our third step, we construct unique diagonalizations. Suppose we are given an epimorphism $e : A \to B$, a source $\{g_i : B \to D_i\}_{i\in I}$, a morphism $f : B \to C$, and an extremal mono-source $\{m_i : C \to D_i\}_{i\in I}$ with the property that $e\,;g_i$ equals $f\,;m_i$ for every i in I. We need to construct a morphism $d : B \to C$ such that $e\,;d$ equals f and $d\,;m_i$ equals f_i for every i in I. Any such morphism is necessarily unique because e is an epimorphism. The mapping $d(b) = c$ is defined to hold whenever $\forall i \in I.\ f_i(b) = m_i(c)$ holds. This mapping is total by induction on the proof that e $\mathsf{generates}^{\Delta}_{\{R_B\}_{R:n\in\Phi}}$ b for all b in B, using the fact that $\{f_i\}_{i\in I}$ is relation-preserving and $\{m_i\}_{i\in I}$ is extremal; the mapping is determined because the relations in C satisfy the determinisms in Δ and $\{m_i\}_{i\in I}$ is a mono-source; and the mapping is relation-preserving because $\{m_i\}_{i\in I}$ is initial. $\square$