Assignment 13

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Definition (Cartesian Closed)**.** Given a category $\mathbf{C}$ with finite products, one can construct a corresponding multicategory with the same objects as $\mathbf{C}$ and with multimorphisms from $[A_1, \ldots, A_n]$ to A' being morphisms of $\mathbf{C}$ from $A_1 \times \cdots \times A_n$ to A'. A category $\mathbf{C}$ is said to be *cartesian closed* if it has finite products and its corresponding multicategory is closed (the corresponding multicategory is always symmetric, in fact cartesian, so left and right closed are equivalent).

More explicitly, a category $\mathbf{C}$ is cartesian closed if it has finite products and for every pair of objects A and B there exists an object E and a morphism $e : A \times E \to B$ such that for every object C and morphism $f : A \times C \to B$ there exists a unique morphism $f' : C \to E$ such that $(A \times f') \,;e$ equals f (where $A \times f'$ is $\langle \pi_1, \pi_2 \,; f' \rangle$). Often the object E is denoted as B^A, the morphism e is called $\mathsf{eval}_{A,B}$, and the induced morphism f' is denoted $\lambda_{A,B,C} f$ (with subscripts often omitted in both cases).

Exercise 1. Given a cartesian-closed category $\mathbf{C}$, prove that the $\mathbf{C}$-indexed category mapping each object I of $\mathbf{C}$ to its simple-slice category $\mathbf{C} /\!/ I$ has simple products. Hint: given a set I and an indexed collection of sets $\{A_i\}_{i \in I}$, the type of a function that maps each element i of I to an element of A_i is often denoted as $\prod_{i \in I} A_i$, but when it happens to be the case that each set A_i is the same regardless of the index i, meaning there is some set A such that A_i equals A for all i in I, then the type of a function is simply denoted as $I \to A$ or A^I.

In order to focus on the most interesting aspects of the proof, just give the definition of $\prod_{I,J}$ on objects of $\mathbf{C} /\!/ (I \times J)$ and give the transposition between $\mathbf{C}(\pi_{I,J})$ and $\prod_{I,J}$. Show that this transposition is bijective, but do not show naturality of the transposition nor naturality of $\prod_{I,J}$ with respect to I.

Definition. A concrete category $\mathbf{A} \xrightarrow{U} \mathbf{X}$ is called $\mathcal{M}$-topological, for a collection of $\mathbf{X}$-sources $\mathcal{M}$, if every structured source in $\mathcal{M}$ has a unique initial lifting.

Example. In the case where $\mathcal{M}$ is all sources in $\mathbf{X}$, then this is simply the definition of a topological category. In the case where $\mathcal{M}$ is all mono-sources in $\mathbf{X}$, then this is known as monotopological.

Exercise 2. Given a $\mathcal{M}$-topological concrete category $\mathbf{A} \xrightarrow{U} \mathbf{X}$, prove that if $\mathbf{X}$ has an $(\mathcal{E}, \mathcal{M})$-factorization structure then $\mathbf{A}$ has an $(\mathcal{E}_U, \text{Initial } \mathcal{M}_U)$-factorization structure, where the collection $\mathcal{E}_U$ is the collection of $\mathbf{A}$-morphisms whose underlying $\mathbf{X}$-morphism is in $\mathcal{E}$, and where the collection Initial $\mathcal{M}_U$ is the collection of initial $\mathbf{A}$-sources whose underlying $\mathbf{X}$-source is in $\mathcal{M}$.

Definition. Let Φ be a collection of "abstract symbols" each paired with a natural number n indicating the arity of the symbol. For example, $\Phi = \{\leq : 2\}$ has one abstract symbol, $\leq$, of arity 2, indicating that there should be a relation $\leq$ that is binary.

Given a Φ, the construct $\mathbf{Rel}(\Phi)$ is comprised of the following:

Object An object is a set A along with, for each symbol-arity pair $R : n \in \Phi$, a relation $R_A \subseteq A^n$.

Morphism A morphism from $\langle A, \{R_A\}_{R:n\in\Phi} \rangle$ to $\langle B, \{R_B\}_{R:n\in\Phi} \rangle$ is a function $f : A \to B$ such that, for every symbol-arity pair $R : n \in \Phi$, given any n-tuple $\vec{a}$ in R_A, the n-tuple $f(\vec{a})$ is in R_B (where $f(\vec{a})$ is shorthand for the n-tuple $\langle f(a_i) \rangle_{i \in \{1,\ldots,n\}}$).

Example. The construct $\mathbf{Rel}(\leq : 2)$ is more explicitly comprised of the following:

Object An object is a set A along with a binary relation $\leq_A \subseteq A \times A$.

Morphism A morphism from $\langle A, \leq_A \rangle$ to $\langle B, \leq_B \rangle$ is a function $f : A \to B$ such that, for all a_1 and a_2 in A, if $a_1 \leq_A a_2$ holds then $f(a_1) \leq_B f(a_2)$ holds.

That is, $\mathbf{Rel}(\leq : 2)$ is simply $\mathbf{Rel}(2)$, and $\mathbf{Rel}(\Phi)$ is a generalization of $\mathbf{Rel}(2)$ that enables both multiple relations and the ability for each relation to have its own arity.

Remark. The construct **Rel**(Φ) is topological. Given a structured source $\{A \xrightarrow{f_i} U(\langle B_i, \{R_i\}_{R:n\in\Phi}\rangle)\}_{i\in I}$, its unique initial lifting is given by the object $\langle A, \{R_A\}_{R:n\in\Phi}\rangle$, where R_A for each $R : n$ in Φ is defined as the subset $\{\vec{a} \in A^n \mid \forall i \in I.\, f_i(\vec{a}) \in R_i\}$. Since **Set**, and in fact every category, has an (Iso, Source)-factorization structure, by your proof above we know that **Rel**(Φ) has an (Iso_U, Initial Source_U)-factorization structure. Furthermore, **Rel**(Φ) has initial liftings of U-structured isomorphisms, and so we can apply the following lemma to **Rel**(Φ).

Lemma. *Given any concrete category* $\mathbf{A} \xrightarrow{U} \mathbf{X}$ *with an* ($Iso_U, \mathcal{M}$)*-factorization structure for some collection of sources* $\mathcal{M}$*, and with unique initial liftings of* U*-structured isomorphisms, any implicational subcategory of* **A** *formed by a collection of identity-carried implications, meaning* **A***-morphisms whose underlying* **X***-morphisms are identities, is concretely reflective over* **X**.

Proof. Because all of the implications are identity-carried, they all belong to Iso_U. Consequently, due to the assumption that **A** has an ($\mathrm{Iso}_U, \mathcal{M}$)-factorization structure, we know that the implicational subcategory is at least Iso_U-reflective. Given an object A of **A**, let $r : A \to A'$ be an Iso_U-reflection arrow into the implicational subcategory. This means that $Ur : UA \to UA'$ is an isomorphism, and clearly it is furthermore U-structured, so by assumption it has a unique initial lifting. Let $s : A'' \to A'$ be the initial lifting of Ur. This means that Us equals Ur, which implies that $Uid_{A'}$ equals $(Ur)^{-1} ; Us$, and so initiality of s implies there exists a morphism $s' : A' \to A''$ such that Us' equals $(Ur)^{-1}$. This s' is in fact the inverse of s since, by faithfulness of U, the fact that they have opposite domain and codomain and have underlying morphisms that are inverses of each other implies that they are themselves inverse of each other. It is easy to show that implicational subcategories are isomorphism dense, implying that s' is furthermore an isomorphism contained in the implication subcategory. Reflection arrows are essentially unique, and so composing the reflection arrow r with the isomorphism s' contained in the implication subcategory necessarily results in a reflection arrow. Furthermore, the underlying morphisms of r and s' are inverses of each other, so the underlying morphism of $r ; s'$ is an identity morphism, making $r ; s'$ an identity-carried reflection arrow for A into the implicational subcategory. Since such an identity-carried reflection arrow exists for all objects of **A**, the implicational subcategory is by definition concretely reflective over **X**. ☐

Exercise 3. Let Φ be the $\mathbb{R}^{\geq}$-indexed collection $\{d_\delta : 2\}_{\delta\in\mathbb{R}^{\geq}}$, where the intention is to view a relation $d_\delta(x, x')$ as indicating that the distance from point x to point x' is at most δ. Using the fact that the lemma above applies to **Rel**(Φ), show that **LMet** is concretely isomorphic to a full concretely reflective subcategory of **Rel**(Φ) over **Set**. More specifically, show the following, as the rest of the proof is mostly tedious (if the implications are correct):

1. Give a collection of identity-carried implications whose corresponding implicational subcategory happens to be concretely isomorphic to **LMet**.
2. Give the corresponding propositional formulations of those implications, e.g. $\forall x, x'.\, x \approx x' \implies x' \approx x$
3. Given an object of **Rel**(Φ) in the implicational subcategory, define the corresponding distance function on its underlying set.
4. Given an object of **LMet**, define the corresponding object of **Rel**(Φ) that happens to be in the implicational subcategory.
5. Given an object A of **Rel**(Φ) in the implicational subcategory, prove that converting it into a distance function and then back into an object of **Rel**(Φ) results in the original object A. You may assume the correspondence between your identity-carried implications and propositional formulations has been justified.

Remark. Note that **Rel**(Φ) is also *mono*topological, since every mono-source is a source, and its underlying category **Set** has an (Epi, Mono-Source)-factorization structure. Consequently, your earlier proof also indicates that **Rel**(Φ) has an (Epi_U, Initial Mono-Source_U)-factorization structure. Any identity-carried implication also belongs to Epi_U, and so your above proof implies that **LMet** is a full Epi_U-reflective subcategory of **Rel**(Φ). This in turn implies that **LMet** has an ($\mathrm{Epi}_{I\,;\,U}$, Initial Mono-$\mathrm{Source}_{I\,;\,U}$)-factorization structure. But since **LMet** is a concrete subcategory, $I ; U$ equals U (where the former U is for **Rel**(Φ) and the latter U is for **LMet**), and so it has an (Epi_U, Initial Mono-Source_U)-factorization structure. And furthermore, **LMet** has both free objects (given by making the distance between any two points 0) and cofree objects (given by making the distance between any two points ∞), and so its epimorphisms and mono-sources coincide with those of the underlying category. Thus, putting everything together, you have effectively proven that **LMet** has an (Epi, Initial Mono-Source)-factorization structure, the most epic factorization structure possible.