CS 6110 Lecture 34 | Recursive Types | 23 April 2025
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1 Introduction

Industrial-strength programming languages have a feature we haven't looked at yet: recursive types. For example, in Java we can define a binary tree like this:

```
class Tree {
 Tree left, right;
 int value;
}
```

Similarly, in ML we can define recursive variants (aka datatypes), with declarations like the following:

```
type tree = Leaf | Node of tree * tree * int
```

In the simply typed-lambda calculus, we have no corresponding mechanism.

Intuitively, we would like a tree type that satisfies the following equation:

tree = 1 + tree * tree * int

The tree on the right-hand side can be replaced by the definition itself and in this way we can get a binary tree of any size. The 1 permits an empty tree to be represented using null, just as we might in Java.

2 Recursive Types as Regular Labeled Trees

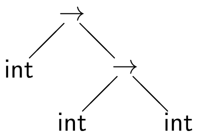
By expanding the equation for tree, we can see that

$$\begin{aligned}
\alpha &= 1 + \alpha * \alpha * \mathsf{int} \\
&= 1 + (1 + \alpha * \alpha * \mathsf{int}) * (1 + \alpha * \alpha * \mathsf{int}) * \mathsf{int} \\
&= 1 + (1 + (1 + \alpha * \alpha * \mathsf{int}) * (1 + \alpha * \alpha * \mathsf{int}) * \mathsf{int}) * (1 + (1 + \alpha * \alpha * \mathsf{int}) * (1 + \alpha * \alpha * \mathsf{int}) * \mathsf{int}) * \mathsf{int} \\
&= \cdots
\end{aligned}$$

At each level, we have a finite type with the type variable α appearing at some of the leaves, and we obtain the next level by substituting the right-hand side of the equation for α. This gives a sequence of deeper and deeper finite trees, where each successive tree is a substitution instance of the previous tree.

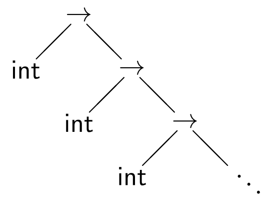
If we take the limit of this process, we have an infinite tree. We can think of this as an infinite labeled graph whose nodes are labeled with the type constructors *, +, int, and 1. This is very much like an ordinary type expression, except that it is infinite. There are no more α's, because we have substituted for all of them all the way down. This infinite tree is a solution of the tree equation, and this is what we take as the type tree.

In general, let Σ be a signature consisting of several type constructors of various arities. For example, Σ might consist of the type constructors $\rightarrow$, *, +, 1, and int. We can form the set of (finite) types over Σ inductively in the usual way. Each such type can be regarded as a finite labeled tree. For example, the type $\mathsf{int} \rightarrow \mathsf{int} \rightarrow \mathsf{int}$ can be viewed as the labeled tree



Now let us add some infinite types. These are infinite labeled trees that respect the arities of the constructors in Σ; that is, if the constructor is binary (such as * or $\rightarrow$), any node labeled with that constructor must have exactly two children; and if the constructor is nullary, such at 1, then any node labeled with that symbol must be a leaf. Within these constraints, the tree may be infinite.

A (finite or infinite) expression with only finitely many subexpressions up to isomorphism is called *regular*. For example, the infinite type



is regular, since it has only two subexpressions up to isomorphism, namely itself and int. The limit of the unrolling of the equation above, which we took to be the type tree, is also regular; it has exactly five subexpressions up to isomorphism, namely tree, 1, tree * tree * int, tree * tree, and int.

Regular trees are all we need to provide solutions to finite systems of type equations. Suppose we have n type equations in n variables:

$$\begin{aligned} \alpha_1 &= \tau_1 \\ &\vdots \\ \alpha_n &= \tau_n, \end{aligned} \tag{1}$$

where each τ_i is a finite type over the type constructors Σ and type variables $\alpha_1, \ldots, \alpha_n$. This system has a solution $\sigma_1, \ldots, \sigma_n$ in which each σ_i is a regular tree. Moreover, if no right-hand side is a variable, then the solution is unique.

3 The μ Constructor

The definition of tree is recursive and not surprisingly we need to take some sort of fixed point – fixed point on types. So we define a function F whose fixed point we need to find (we use a double colon to show that it's a function operating on types):

$$F(X) = \lambda\alpha :: \mathit{type}.\ 1 + \alpha * \alpha * \mathit{int}$$

$$\mathbf{tree} = F(\mathbf{tree})$$

Let us posit a *fixed-point type constructor* that solves these equations. If $F(\alpha)$ is the function whose fixed point we are trying to find, we denote the fixed point as $\mu\alpha.\ F(\alpha)$. By definition, $F(\mu\alpha.\ F(\alpha)) = \mu\alpha.\ F(\alpha)$. So we can use this construct to define $\mathsf{tree} = \mu\alpha.\ 1 + \alpha * \alpha * \mathsf{int}$. If we write τ for $F(\alpha)$, then fixed point would be $\mu\alpha.\ \tau$. Thus,

$$F(\mu\alpha.\ \tau) = \tau\{\mu\alpha.\ \tau/\alpha\} = \mu\alpha.\ \tau$$

This step of substituting the fixed point for α inside τ "unfolds" τ. Going from one side to another of the following equality we get the **fold** and **unfold** operations.

$$\mu\alpha.\ \tau = \tau\{\mu\alpha.\ \tau/\alpha\}$$

$$\mathsf{unfold} : \mu\alpha.\ \tau \longrightarrow \tau\{\mu\alpha.\ \tau/\alpha\}$$

$$\mathsf{fold} : \mu\alpha.\ \tau \longleftarrow \tau\{\mu\alpha.\ \tau/\alpha\}$$

We require the type $\mu\alpha.\tau$ must be *contractive*: the expression τ must use some type constructor and cannot be simply another type variable β. This condition means that applying fold makes the type smaller and unfold makes it larger. For example, the type $\mu\alpha.\ \alpha$ is illegal because it is not contractive.

The solutions $\sigma_1, \ldots, \sigma_n$ to any finite system of the form (1) can be expressed in terms of μ. For example, suppose τ_1 and τ_2 are finite type expressions over the type variables α_1, α_2 such that neither τ_1 nor τ_2 is a variable. The system

$$\alpha_1 = \tau_1 \qquad\qquad \alpha_2 = \tau_2$$

has a unique solution σ_1, σ_2 specified by

$$\sigma_1 = \mu\alpha_1.\,(\tau_1\{\mu\alpha_2.\,\tau_2/\alpha_2\}) \qquad\qquad \sigma_2 = \mu\alpha_2.\,(\tau_2\{\mu\alpha_1.\,\tau_1/\alpha_1\}).$$

Mutually recursive type declarations arise quite often in practice. For example, consider the following Java class definitions for Node and Edge:

```
class Node {
   Edge[] inEdges, outEdges;
}

class Edge {
   Node source, sink;
}
```

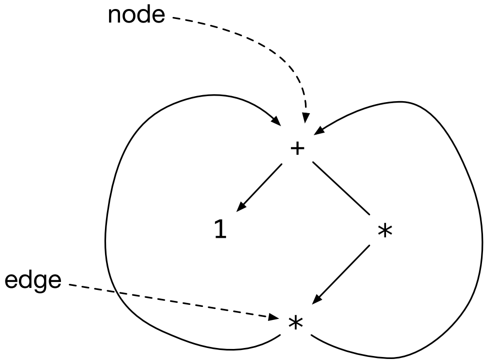
Note that Node refers to Edge and vice versa. So we must take a mutual fixed point when assigning a meaning to such types. Writing these types in the style of the typed lambda calculus, we have these equations:

$$\begin{aligned}\mathsf{node} &= 1 + \mathsf{edge} * \mathsf{node}\\ \mathsf{edge} &= \mathsf{node} * \mathsf{node}\end{aligned}$$

Following the recipe above, the solutions to these equations are as follows:

$$\begin{aligned}\mathsf{node} &= \mu n.\, 1 + (\mu e.\, n * n) * n\\ \mathsf{edge} &= \mu e.\, ((\mu n.\, 1 + e * n) * (\mu n.\, 1 + e * n)\end{aligned}$$

It is perhaps more useful (and certainly more compact) to view these types as graphs rather than the infinite trees they unfold into. The uses of the type variables n and e correspond to back edges in the graph.



4 Isorecursive vs. Equirecursive types

When we write programs in Java or ML, we never write this μ construct. Instead the languages do this for us. In most programming languages this equality is an isomorphism rather than an equality:

$$\mu\alpha.\, \tau \cong \tau\{\mu\alpha.\, \tau/\alpha\}$$

Because of this isomorphism, we have to change the views through fold and unfold operations. Based on how the conversion is handled by a programming language, we get two approaches to recursive types:

- **Iso-recursive types** where $\mu\alpha.\, \tau$ and $\tau\{\mu\alpha.\, \tau/\alpha\}$ are isomorphic but not equal, and the fold and unfold operations are explicit. Most programming languages handle it roughly in this way, but by combining fold and unfold with an existing operation that is supported by the kind of type that supports recursion.

  For example, ML has recursive variants, but a type is not equal to its unfolding. The case (or match) construct signals to the compiler that a recursive datatype needs to be unfolded. The constructors for datatypes, conversely, implicitly fold the type.

  Conversely, Java supports type recursion through class types, which are roughly records. It unfolds field types when the fields are selected and folds them when objects are constructed.

- **Equi-recursive types** where either there are no operations indicating when to fold and unfold recursive types. A few languages do support this, such as Modula-3. For example, in Modula-3, the following two types are interchangeable in any context:

```
 TYPE foo =  RECORD x:  INTEGER, y:  REF foo  END
 TYPE bar =  RECORD x:  INTEGER,
                  y:  REF RECORD
                   x:  INTEGER,
                   y:  REF bar
                  END
         END
```

  It is possible for the Modula-3 compiler to efficiently decide equality of types foo and bar because the recursive type definitions stand for regular types. Therefore only a finite number of type expressions need to be compared.

5 Typing Rules

Let's explore the more explicit isorecursive approach. We can write the typing rules for fold and unfold. Like many of the rules we have seen so far, these will simply be a pair of "introduction" and "elimination" rules for μ-types.

$$\frac{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}{\Gamma \vdash \mathsf{fold}_{\mu\alpha.\,\tau}\ e : \mu\alpha.\,\tau}\ (\mu \text{ introduction})$$

$$\frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash \mathsf{unfold}\ e : \tau\{\mu\alpha.\,\tau/\alpha\}}\ (\mu \text{ elimination})$$

Suppose we want to type-check a fold expression (annotated so we know what the fold is meant to do). We can think of it as function that gives you a $\mu\alpha.\,\tau$, given that its argument e has type $\tau\{\mu\alpha.\,\tau/\alpha\}$, which is really just the same type. This is what we have in the first rule.

The second rule simply says that unfold will do the opposite. If e has type $\mu\alpha.\,\tau$, then unfold e has type $\tau\{\mu\alpha.\,\tau/\alpha\}$.

These two rules say that fold and unfold behave just like functions as we have described them.

6 An Example

We can see how recursive types work by writing some code. Suppose we want to write a program to add up a list of numbers. How would we define a recursive list type? It's a recursive type, so there'll be a $\mu\alpha.$. We'll need a 1 to represent the empty list, and the general case is that we have an int followed by the rest of the list, i.e., $\mathsf{int} * \alpha$. So we can define

$$\mathsf{intlist} \ \triangleq\ \mu\alpha.\,1 + \mathsf{int} * \alpha$$

Now we can write our function to add up an intlist, which we'll call sum. This is going to be a recursive function, so we'll need to take a fix point and declare it as:

$$\mathsf{let\ sum} = \mathsf{rec}\ f : \mathsf{intlist} \to \mathsf{int}.\ \lambda l : \mathsf{intlist}.$$

And what do we do in the body of this function? We want to do a case on l, but we need a sum, and l is a μ-type. So we need to first unfold l (and this is what ML will do for you automatically when it sees a case). So we have

$$\mathsf{let}\ l' : 1 + \mathsf{int} * \mathsf{intlist} = \mathsf{unfold}\ l\ \mathsf{in}$$

And now we can do our typed case.

$$\begin{array}{ll} \textsf{case}\ \ l'\ \textsf{of} & \\ & u{:}\,1.\ 0 \\ \mid & p{:}\,\textsf{int} * \textsf{intlist}.\ (\#1\ p) + f\ (\#2\ p) \end{array}$$

This is just the same code that you would write in ML, except you can see explicitly some things that ML hides from you. In particular, we've explicitly shown that there is recursion happening with our definition of the intlist type, and the unfold that needs to happen to get the different views of our type.

7 Structural Operational Semantics

The operational semantics are straightforward. We have our introduction form (fold), and our elimination form (unfold). Both of them are eager:

$$E ::= \ldots \ \mid \textsf{fold}_{\mu\alpha.\,\tau}\ E \mid \textsf{unfold}\ E$$

By now, you know that all structural operational semantics is just the elimination form wrapped around the introduction form and some magic happens. So the left hand side would be unfold $(\textsf{fold}_{\mu\alpha.\,\tau}\ v)$. And what does this step to? No surprise: just v.

$$\textsf{unfold}\ (\textsf{fold}_{\mu\alpha.\,\tau}\ v) \longrightarrow v$$

This is showing that there's nothing really interesting happening here with these fold and unfold operations. They are just shifting views, and the fold and unfold mark which view we are looking at.

8 Self-Application and Ω

Recall the self-application and Ω terms that we saw in previous lectures.

$$\begin{array}{rcl} SA & \triangleq & \lambda x.\, x\ x \\ \Omega & \triangleq & SA\, SA \end{array}$$

We can now give these terms types, by adding some judicious folding. We know that x has to be some kind of function type, where it can take itself as its argument. So x must have a type like:

$$x : \mu\alpha.\, \alpha \to \tau$$

for some type τ.

So what will be the type of unfold x? We simply do one step of unfold for our μ-type.

$$\textsf{unfold}\ x : (\mu\alpha.\, \alpha \to \tau) \to \tau$$

Now unfold x looks like a function that takes in something of the type of x as an argument. So we can take unfold x and apply it to x. And what's the type of $(\textsf{unfold}\ x)\ x$?

$$(\textsf{unfold}\ x)\ x : \tau$$

since unfold x gives us a τ, and we gave it an appropriate argument.

So now we can write the fully typed SA term.

$$SA \ \triangleq\ \lambda x : \mu\alpha.\, \alpha \to \tau.\, (\textsf{unfold}\ x)\ x : (\mu\alpha.\, \alpha \to \tau) \to \tau$$

What if we folded the SA type? We unfolded x from $\mu\alpha.\, \alpha \to \tau$ to $(\mu\alpha.\, \alpha \to \tau) \to \tau$, so we can fold back in the opposite direction:

$$\mathsf{fold}_{\mu\alpha.\,\alpha\to\tau}\ SA : \mu\alpha.\,\alpha \to \tau$$

Therefore, we can take SA and apply it to fold SA, and get a τ.

$$SA\ (\mathsf{fold}\ SA) : \tau$$

And this, you'll notice, is just the same as the term Ω. If we run our operational semantics, we discover that this expression goes into an infinite loop, just like before. And yet, it's well-typed.

We never chose a type τ, and indeed, it will type-check no matter what τ is—a sure sign that the term will never produce any result.

$$\Omega : \tau$$

9 Numbers as a Recursive Type

What else can we do with recursive types? Turns out we don't need to have natural numbers anymore. Recall that the typed λ-calculus started out with 1, boolean, and int. We already saw how to get rid of boolean, by translating it to the sum $1 + 1$.

Using recursive types, we can encode numbers. A natural number is either 0, which we will represent as null, or it's the successor of a natural number. Thus, we can define a type for natural numbers:

$$\mathsf{nat} \ \triangleq\ \mu\alpha.\, 1 + \alpha$$

Our representation of 0 is a folded *null* that has been injected into the sum. And *unit* would is folded 0, and so on.

$$\begin{array}{rcl} 0 & \triangleq & \mathsf{fold}_{\mathsf{nat}}\ \mathsf{inl}(null) \\ 1 & \triangleq & \mathsf{fold}_{\mathsf{nat}}\ \mathsf{inr}(0) \\ 2 & \triangleq & \mathsf{fold}_{\mathsf{nat}}\ \mathsf{inr}(1) \\ & \ldots & \end{array}$$

We can use nat to code up all of the usual arithmetic that we want. For example, successor would be

$$\mathit{SUCC} \ \triangleq\ \lambda x : \mathsf{nat}.\, \mathsf{fold}_{\mathsf{nat}}\ \mathsf{inr}(x) : \mathsf{nat}$$

So all we really need is the 1 type. If we have recursive types, products, and sums, we can build all of the other types like natural numbers, integers, floating point numbers, and so on from the 1 type.

10 Untyped to Typed λ-Calculus

Now that we have all the expressive power we want in types, we can encode the (untyped) λ-calculus with types. Taking an arbitrary λ-calculus term that does not say what its type is, we can write down a type for it. So what's the type of a term in the λ-calculus?

Let's call that type U. We know that terms in the λ-calculus are all functions that can take in other terms of the λ-calculus and give you back another term in the λ-calculus as a result. So it must be the case that U is isomorphic to U $\to$ U. That means we can define it as:

$$\mathsf{U} \ \triangleq\ \mu\alpha.\, \alpha \to \alpha$$

which will satisfy the isomorphism.

And we can in fact write a translation from the λ-calculus to the typed λ-calculus with recursive types.

$$\begin{aligned}
[\![x]\!] &= x\\
[\![e_0\, e_1]\!] &= (\mathsf{unfold}\ e_0)\ e_1\\
[\![\lambda x.\, e]\!] &= \mathsf{fold}_{\mathsf{U}}\ \lambda x : \mathsf{U}.\, [\![e]\!]
\end{aligned}$$

Note that every translation gives us something of type U.

This translation can be implemented directly in SML and OCaml by defining recursive types.

```
type u = Fun of f
and f = u->u
...
```

Then fold is implemented using the Fun constructor, and unfold is implemented using case/match.

11 Checking equivalence of equirecursive types

In a language with equirecursive types, the type checker effectively folds and unfolds recursive types automatically as needed rather than requiring some explicitly operation to indicate where it should happen. This behavior can be characterized by adding two new subsumption rules to the type system:

$$\frac{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}{\Gamma \vdash e : \mu\alpha.\,\tau} \qquad \frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}$$

If the language has subtyping, we could alternatively simply make the folded and unfolded types subtypes of each other and rely on the existing subsumption rule.

However, as with subtyping, adding these subsumption rules makes the type system no longer syntax-directed. The solution is the same: we construct a syntax-directed type system whose proofs correspond to normalized proofs of the original type system.

To construct such a type system, we introduce a judgment $\vdash \tau_1 \cong \tau_2$ and fold its use into each of the typing rules, just like we did for subtyping. For example, the rule for typing applications incorporates possible syntactic disagreement between the argument and parameter types:

$$\frac{\Gamma \vdash e : \tau'' \to \tau \quad \Gamma \vdash e' : \tau' \quad \vdash \tau' \cong \tau''}{\Gamma \vdash e\, e' : \tau}$$

How do we decide whether two types are equivalent? Since recursive types really represent regular trees, two types are equivalent if their infinite unfoldings are equivalent. Unfortunately, this insight does not lead immediately to a simple proof system. Intuitively, we might think we could write rules like the following:

$$\frac{\vdash \tau_1 \cong \tau_1' \quad \vdash \tau_2 \cong \tau_2'}{\vdash \tau_1 \to \tau_2 \cong \tau_1' \to \tau_2'} \qquad \frac{\vdash \tau\{\mu\alpha.\,\tau/\alpha\} \cong \tau'}{\vdash \mu\alpha.\,\tau \cong \tau'}$$

However, these rules do not allow us to prove simple equivalences. For example, if we define $U = \mu\alpha.\,\alpha \to \alpha$, we expect equivalences like $U \cong U$ and $U \cong U \to U$. Using the proof rules above, attempting to prove either of these equivalences causes us to prove the other. The proof never bottoms out in a axiom. Proving equality of infinite trees with this proof system requires infinite proof trees; that is, we actually want the *greatest* fixed point of the proof rules rather the usual least fixed point. That is, we are doing coinductive reasoning.

One approach to solving this equivalence problem is to view the type as defining a regular automaton, with back edges at the points where a type variable is mentioned. This automaton corresponds to the graphs shown earlier for the node and graph types, with the edges labeled according to the types appearing at each node. For example, the two edges from a * node might be labeled *1 and *2 to distinguish them.

A second and equivalent approach is to design an inductive type system that can derive all the judgments that the greatest-fixed-point solution would. The intuition is that once we have tried to prove some judgment at a particular point in the proof tree, we can assume that judgment holds above that point in the proof tree. If the whole (infinite) proof tree can be contructed under the assumption that the assumed judgment can be proved, then a copy of the proof tree can be used to derive the assumption! We assume an environment E for recording type-equivalence assumptions, and write $E \vdash \tau_1 \cong \tau_2$ to mean that types τ_1 and τ_2 are equivalent, assuming that all equivalences in E hold:

$$E ::= \tau_1 \cong \tau_1', \ldots, \tau_n \cong \tau_n' \ ^{(n \geq 0)}$$

$$\frac{}{\tau_1 \cong \tau_2 \vdash \tau_1 \cong \tau_2} \qquad \frac{\tau_1 \cong \tau_1' \quad \tau_2 \cong \tau_2'}{\tau_1 \to \tau_2 \cong \tau_1' \to \tau_2'}$$

The key rule is the one that unfolds a recursive type but also adds an assumption (symmetrically, we can unfold the right-hand side):

$$\frac{E, \mu\alpha.\, \tau \cong \tau' \vdash \tau\{\mu\alpha.\, \tau/\alpha\} \cong \tau'}{E \vdash \mu\alpha.\, \tau \cong \tau'}$$

For this proof system to work, it is important that recursive types be contractive. Otherwise $\mu\alpha.\, \alpha$ would be considered equivalent to every other type.

12 Semantics of recursive types

One question we can ask is what fixed point we expect to get from the μ type constructor. One way to solve these equations is to consider them as inductive definitions, in which the type corresponds to the union of all sets generated by finite applications of type constructors. This is an adequate interpretation for eager languages where we do not expect to be able to construct infinite trees by taking fixed points. However, even in an eager language, we expect to be able to take fixed points over functions. So for general F, we want the type to denote CPOs over which we can take fixed points. The inductive construction will not do that, and we will need some additional machinery for a construction that will.

13 Closed vs Open Recursion

There is one thing that is limiting about our take on recursive types so far. A type like $\mu\alpha.\, \alpha \to \alpha$ is a *closed* recursion mechanism, in which the scope of the recursion is very clearly defined. There is a $\mu\alpha.$, and α can only be mentioned in its body, and you therefore know exactly where you are taking a fixed point. That's nice, since the meaning of types is locally defined in some sense.

Many languages, such as Java, provide an open recursion mechanism. For example, in Java, a class can refer to other classes, which can refer freely back to the original class. This is nice because you don't have to define an ordering on classes:

```
class A {
  B x;
}

class B {
  A x;
}
```

Notice that nowhere did we have to say that there's a fixed point. Teaning of a bunch of possibly mutually recursive types in Java is really a big fixed point over all of the classes in the system, since they can all refer to each other in arbitrarily complicated ways. With open recursion mechanisms, different names in the program can implicitly correspond to fixed point constructions; the meaning of names is not defined in a local way. In Java, the names A and B refer to particular components of a big fixed point taken implicitly over the entire system.

The benefit of this approach is that it supports extensibility and reuse. You can grab a bunch of classes that you have sitting around, pull them into a system, and you don't have to define where the fixed point is taken. In fact, the classes taken off the shelf can even refer to classes that are not yet defined, to be supplied by you. The fixed point will then be automatically taken across the additional classes.

Open recursion gives you a lot of flexibility, but it does come with a price. There are some semantic issues. Suppose we have the following.

```
class A {
  static final int x = B.x + 1;
}

class B {
  static final int x = A.x + 1;
}
```

What does this code mean? In Java lexicon, `static final` fields are supposed to look like compile-time constants. But there is no actual fixed point solution to this code. There are no integers that we can assign to `A.x` and `B.x` that can make both of the equations come out true. So open recursion mechanisms are a bit problematic.

What do you get if you actually do this in Java? You'll get a 1 and a 2, but which is which depends on the order that these two classes are initialized.