There are a few alternative characterizations of CPOs and continuity that are useful to know about. In particular, the notion of continuity as presented in Lecture 7 is equivalent to continuity in the traditional topological sense with respect to a certain topology induced by the order. This topology is called the *Scott topology* after Dana Scott (1932–).

## 1 CPOs and Directed Sets

First we give an alternative characterization of CPOs in terms of a seemingly stronger notion of completeness.

**Definition C.1.** *Let $(X, \sqsubseteq)$ be a poset. A subset $A \subseteq X$ is said to be* directed *if for every pair of elements $a, b \in A$, there exists $c \in A$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. The poset $X$ is said to be* directed-complete *if every directed subset of $X$ has a supremum in $X$.*

**Theorem C.2.** *A poset is chain-complete (that is, it is a CPO) iff it is directed-complete.*

*Proof.* Clearly every directed-complete poset is chain-complete, since chains are directed sets. Thus every directed-complete poset is a CPO.

For the converse, suppose $X$ is chain-complete. We will construct for every directed subset $A$ a certain chain $C$ in $X$ and show that its supremum is also the supremum of $A$. The construction is by transfinite induction on cardinality.

Let $A$ be a directed subset of $X$. Then every finite subset $F$ of $A$ has an upper bound $b_F \in A$. Moreover, by constructing $b_F$ inductively, we can ensure that $b_{\{x\}} = x$ and if $F \subseteq F'$, then $b_F \sqsubseteq b_{F'}$.

If $A$ is finite, then its supremum is $b_A$ and we are done. Otherwise, let $\delta = |A|$ be the cardinality of $A$ (the least ordinal in one-to-one correspondence with $A$). Enumerate $A$ as $A = \{x_\alpha \mid \alpha < \delta\}$. For each $\beta < \delta$, let

$$A_\beta = \{b_F \mid F \text{ is a finite subset of } \{x_\alpha \mid \alpha < \beta\}\}.$$

Here are some things to observe about the $A_\beta$:

(i) $A_\beta$ is directed: Every $b_F, b_G \in A_\beta$ have an upper bound $b_{F \cup G} \in A_\beta$.

(ii) If $A$ is infinite, then $|A_\beta| < |A|$. This is because $A_n$ is finite for finite $n$, and for infinite $\beta < \delta$, the set of all finite subsets of $\{x_\alpha \mid \alpha < \beta\}$ has the same cardinality as $\{x_\alpha \mid \alpha < \beta\}$, which is the cardinality of $\beta$, which is less than $\delta$.

(iii) The $A_\beta$ form a chain with respect to set inclusion: if $\beta \leq \gamma$, then $A_\beta \subseteq A_\gamma$.

(iv) $A = \bigcup_{\beta<\delta} A_\beta$, since every $x_\alpha \in A$ is $b_{\{x_\alpha\}} \in A_{\alpha+1}$.

Thus the sets $A_\beta$ for $\beta < \delta$ form a chain of directed sets of smaller cardinality than $A$ whose union is $A$. By the induction hypothesis, each $A_\beta$ has a supremum $\bigsqcup A_\beta \in X$ (note: not necessarily in $A$!). Let $C = \{\bigsqcup A_\beta \mid \beta < \delta\}$. Then $C$ is a $\sqsubseteq$-chain by (iii) above: if $\beta \leq \gamma$, then $A_\beta \subseteq A_\gamma$, therefore $\bigsqcup A_\beta \sqsubseteq \bigsqcup A_\gamma$.

Since $X$ is chain-complete, $C$ has a supremum $\bigsqcup C$. We claim that this is also the supremum of $A$. It is an upper bound for $A$, since for every $x_\alpha \in A$, $x_\alpha \sqsubseteq \bigsqcup A_{\alpha+1} \sqsubseteq \bigsqcup C$. But any upper bound $x$ for $A$ is also an upper bound for $A_\beta$, $\beta < \delta$, so $\bigsqcup A_\beta \sqsubseteq x$. As this is true for every $\beta < \delta$, $x$ is an upper bound for $C$, therefore $\bigsqcup C \sqsubseteq x$. $\square$

**Theorem C.3.** *Let $D$ and $E$ be CPOs. A monotone function $f : D \to E$ is continuous iff it preserves suprema of directed sets.*

*Proof.* Since $f$ is monotone, the image of a chain is a chain and the image of a directed set is a directed set. If $f$ preserves suprema of directed sets, then it preserves suprema of chains, since every chain is a directed set. For the converse, we can again proceed by transfinite induction on cardinality as in the proof of Theorem C.2. Let $A$ be directed and let $A_\beta$ and $C$ be as in that proof. Then $\bigsqcup A = \bigsqcup C$, $f(x) \sqsubseteq f(\bigsqcup A)$ for $x \in A$, and by the induction hypothesis, $f(\bigsqcup A_\beta) = \bigsqcup_{x \in A_\beta} f(x)$, therefore

$$f(\bigsqcup A) = f(\bigsqcup C) = \bigsqcup_{y \in C} f(y) = \bigsqcup_{\beta < \delta} f(\bigsqcup A_\beta) = \bigsqcup_{\beta < \delta} \bigsqcup_{x \in A_\beta} f(x) = \bigsqcup_{x \in A} f(x). \qquad \square$$

The proofs of Theorems C.2 and C.3 are taken from [2].

## 2 The Scott Topology

A *topological space* is a pair $(X, \tau)$, where $X$ is a set and $\tau$ is a *topology* on $X$, a collection of subsets of $X$ containing $X$ and $\varnothing$ and closed under finite intersections and arbitrary unions. The elements of $\tau$ are called the *open sets* of the topology. A subset of $X$ is said to be *closed* if its complement is open.

The usual notion of continuity for functions between topological spaces is that the inverse image of any open set is open. That is, for functions $f : (X, \tau) \to (Y, \sigma)$ and $A \subseteq Y$, let $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. The function $f$ is said to be *continuous* if $f^{-1}(A) \in \tau$ whenever $A \in \sigma$. Equivalently, $f$ is continuous if the inverse image of any closed set is closed.

There is a natural topology on any CPO $(X, \sqsubseteq)$, the *Scott topology*, induced by the ordering relation $\sqsubseteq$.

**Definition C.4.** *Let $(X, \sqsubseteq)$ be a CPO. A subset $C \subseteq X$ is* closed *in the Scott topology* (Scott-closed) *if*

(i) *$C$ is closed downwards under $\sqsubseteq$; that is, if $x \sqsubseteq y$ and $y \in C$, then $x \in C$;*

(ii) *$C$ is closed under suprema of directed sets; that is, if $D$ is a directed set and $D \subseteq C$, then $\bigsqcup D \in C$.*

*Equivalently stated, $A$ is open in the Scott topology* (Scott-open) *if*

(i) *$A$ is closed upwards under $\sqsubseteq$; that is, if $x \sqsubseteq y$ and $x \in A$, then $y \in A$;*

(ii) *if $D$ is a directed set and $\bigsqcup D \in A$, then $D \cap A \neq \varnothing$.*

Clause (ii) in this definition is unchanged if we replace "directed set" by "chain". One can easily show that finite intersections and arbitrary unions of open sets are open, thus the Scott-open sets form a topology.

The following theorem asserts that a function on CPOs is continuous as defined in Lecture 7 iff it is continuous in the traditional topological sense with respect to the Scott topology.

**Theorem C.5.** *Let $D$ and $E$ be CPOs. A monotone function $f : D \to E$ is continuous* (*in that it preserves suprema of chains or directed sets*) *iff $f^{-1}(A)$ is Scott-open in $D$ whenever $A$ is Scott-open in $E$.*

Once set up, this is not difficult to prove.

The Scott topology is weaker than other perhaps more familiar topologies such as those based on metrics $d : X^2 \to \mathbb{R}_+$. Metric topologies are generated by basic open neighborhoods $N_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$

and enjoy strong separation properties, e.g. they are *Hausdorff*: every pair of distinct points are contained in disjoint open sets. Indeed, if $\varepsilon = d(x, y)/2$, then $N_\varepsilon(x) \cap N_\varepsilon(y) = \varnothing$. The Scott topology is not Hausdorff, but only $T_0$, which just says that for every pair of distinct points $x, y$, the set of open sets containing $x$ is not the same as the set of open sets containing $y$. But note that if $x \sqsubseteq y$, then every open set containing $x$ also contains $y$, so the topology cannot be Hausdorff.

## References


[1] Samson Abramsky and Achim Jung. Domain theory. 1994.

[2] G. Markowsky. Chain-complete p.o. sets and Directed Sets with Applications. *Algebra Universalis*, 6:53–68, 1976.