Recursive Number Theory

A Development of Recursive Arithmetic in a Logic-free Equation Calculus

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The fundamental role which primitive recursion plays in the development of the arithmetic of the natural numbers was discovered by Th. Skolem in 1923. In the present account arithmetic is constructed in a formal equation calculus in which the only axioms are recursive function definitions, and the only rules of proof are substitution rules and a rule affirming the uniqueness of a function defined by recursion. Within the framework of this system of arithmetic, formal logic is introduced in terms of an arithmetical model, and has only the ancillary function of revealing connections and facilitating organisation instead of its customary role in the proof process. The construction of the arithmetical model of propositional logic is followed by a systematic study of bounded operators, (universal, existential and minimal) leading to many recursive analogues of the formulae of predicate logic. *Counting* is introduced into the formalism by means of a recursive operator which counts the number of instances when a predicate holds in an assigned range of values of its argument.

In addition to the applications of recursive function theory in the proofs of the fundamental theorem of arithmetic and generalised principles of mathematical induction, there are accounts of the reduction to primitive recursion of various classes of recursions, including simultaneous recursions, course-of-values recursions and recursion with parameter substitution. Recursive arithmetic is a verifiable system whose provable formulae are equations which may be transformed into the equation $0 = 0$ when definite numerals are substituted for the numeral variables. Although every provable formula is verifiable the converse does not hold and by means of Godel's process of arithmetisation it is shown that there is a verifiable equation which is not provable; this incompleteness is not the consequence of an incomplete set of axioms but lies in the nature of formalised arithmetic as we learn from Skolem's discovery that no formalisation of arithmetic is categorical.

To each of the first four chapters is appended a large collection of examples, illustrating the operation of the equation calculus and the arithmetical models of propositional and predicate logic, with detailed solutions at the end of the book.