CS611 Lecture 7 Inductive definitions and rule induction 12 September, 2005

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1 Syntax

A rule instance is of the form: $\frac{x_1\ x_2\ \ldots x_n}{x}$

And given $\mathbf{B} \subseteq \mathbf{S}$, the rule operator R($\mathbf{B}$)=$\{\ x \mid \frac{x_1\ x_2\ \ldots\ x_n}{x}$ is a rule instance and $\{x_1\ x_2\ \ldots\ x_n\} \subseteq \mathbf{B}\}$

The following facts can be easily verified:

- R($\emptyset$)={axioms}
- R(A)∪R(B)⊆ R(A∪B)
- R: $2^S \rightarrow 2^S$ where 2^S=P(S) is the power set of S

Our goal is to find what set $\mathbf{A} \subseteq \mathbf{S}$ is defined by the rules instances. In addition, A should be:

1. Consistent: A⊆R(A)
2. Closed : R(A)⊆A

i.e., A=R(A). In other words, A is a fixed point of the function R. The there are two natural questions we want to ask: 1.*Is there actually a fixed point of R?* 2.*If yes, which one is A?*

2 Definition of set A

We define the set A as: A=$\cup_{n \in N} R^n(\emptyset) = R(\emptyset) \cup R(R(\emptyset))\ldots$

Next, we prove 3 properties of A, namely:

1. A is closed
2. A is consistent
3. A is the least fixed point of R

2.1 A is closed: R(A)⊆A

For any x ∈R(A), we know that there is some rule instance $\frac{x_1\ x_2\ \ldots\ x_n}{x}$ where $\{x_1\ x_2\ \ldots\ x_n\} \subseteq A$. Thus, for some minimum m, we must have $\{x_1\ x_2\ \ldots\ x_n\} \subseteq R^m(\emptyset)$. Then $x \in R^{m+1}(\emptyset) \subseteq$ A

2.2 A is consistent: A⊆R(A)

We need to introduce a concept here: the monotonicity of the rule operator R. We say R is monotonic if and only if A⊆B ⇒R(A)⊆R(B)

So, if x∈ A then x∈ $R^m(\emptyset)$ for some m, i.e. x∈ $R(R^{m-1}(\emptyset))$. Since $R^{m-1}(\emptyset) \subseteq A$, by the monotonicity of R, we have $R(R^{m-1}(\emptyset)) \subseteq$ R(A). Thus x∈R(A) and A is consistent.

2.3 A is the least fixed point

Suppose A is not the least fixed point of R, then there is some B such that B=R(B) and B⊂A.

Since $\emptyset$ ⊆B⇒R($\emptyset$)⊆R(B)=B by the monotonicity of R. Similarly, we will have $R^m(\emptyset)$⊆R(B)=B where m=1,2,3,. . . Then if we union the left-hand side of ⊆ we will get A by definition. And the union of the right-hand side is just B. So A⊆B which is a contradiction. As a result, A must be the least fixed point of R.

3 Rule Induction

We use well founded induction on the sub-derivation relation to prove properties of inference rules.

Note - should there exist more than one derivation, consider only the shortest derivation.

Theorem: $(FV(e) = \emptyset \wedge e \rightarrow e') \Rightarrow (FV(e') = \emptyset)$

Proof: by induction on derivation of $e \rightarrow e'$

Case 1: $\dfrac{e_1 \longrightarrow e_1'}{(e_1\ e_2) \longrightarrow (e_1'\ e_2)}$
Assume: $FV(e_1\ e_2) = FV(e_1) \cup FV(e_2) = \emptyset$, thus $FV(e_1) = FV(e_2) = \emptyset$

We can use the inductive hypothesis $(FV(e_1) = \emptyset \wedge e_1 \rightarrow e_1') \Rightarrow (FV(e_1') = \emptyset)$ to show that that

$$FV(e_1'\ e_2) = FV(e_1') \cup FV(e_2) = \emptyset$$

Case 2: $\dfrac{e_2 \longrightarrow e_2'}{(v\ e_2) \longrightarrow (v\ e_2')}$
This case is symmetric to Case 1 where now e_2 is used in the inductive hypothesis

Case 3: $\dfrac{}{(\lambda x.\, e)v \longrightarrow e\{v/x\}}$
Assume: $FV(\lambda x.\, e)v = \emptyset, \quad FV(v) = \emptyset, \quad FV(\lambda x.\, e) = FV(e) - \{x\} = \emptyset, \quad \text{thus } FV(e) \in \{x\}$
This case requires a lemma (stated below) to show that $FV(e\{v/x\}) = \emptyset$. Once that is shown, the proof for this case is complete.

We have now considered all three rules of derivation for Lambda Calculus to show that $FV(e) = \emptyset$ is preserved as derivation rules are applied to e

□

Lemma: $(FV(v) = \emptyset) \Rightarrow (FV(e\{v/x\}) = FV(e) - \{x\})$ (this lemma is used by Case 3 in the above theorem)

Proof: by induction on derivation of the substitution relation. Note that substitution isn't really a function because there is choice of variable when the body of a lambda abstraction is alpha-renamed.

Case 1: $e = x, \quad \text{then } x\{e/x\} = e$

$$FV(e\{v/x\}) = FV(v) = \emptyset = \{x\} - \{x\} = FV(x) - \{x\}$$

Case 2: $e = y, \quad \text{then } y\{e/x\} = y$

$$FV(e\{v/x\}) = FV(y) = \{y\} = FV(y) - \{x\}$$

Case 3: $e = e_1\ e_2$
Recall that by definition: $FV(e\{v/x\}) = FV(e_1\{v/x\}\ e_2\{v/x\}) = FV(e_1\{v/x\}) \cup FV(e_2\{v/x\})$
By the inductive hypothesis, $FV(e_1\{v/x\}) = FV(e_1) - \{x\}$ and $FV(e_2\{v/x\}) = FV(e_2) - \{x\}$

$$\begin{aligned} FV(e_1\{v/x\}) \cup FV(e_2\{v/x\}) &= (FV(e_1) - \{x\}) \cup (FV(e_2) - \{x\}) \\ &= (FV(e_1) \cup FV(e_2)) - \{x\} \\ &= FV(e_1\ e_2) - \{x\} \end{aligned}$$

Case 4: $e = (\lambda x.\, e')\{v/x\}$

$$FV((\lambda x.\, e')\{v/x\}) = FV(e) = FV(e) - \{x\} \qquad \text{(because } x \notin FV(e)\text{)}$$

Case 5: $e = (\lambda y.\, e')\{v/x\}$ where $y \neq x$.

Note that $y \notin FV(v)$ because $FV(v) = \emptyset$. Therefore we can use the simpler substitution rule for lambda terms.

$$\begin{array}{rcll}
FV((\lambda y.\, e')\{v/x\}) & = & FV(\lambda y.\, e'\{v/x\}) & \\
& = & FV(e'\{v/x\}) - \{y\} & \\
& = & FV(e'\{v/x\}) - \{y\} & \\
& = & FV(e') - \{x\} - \{y\} & \text{(by the inductive hypothesis)} \\
& = & FV(\lambda y.\, e') - \{x\} & \\
& = & FV(e) - \{x\} &
\end{array}$$

$\square$