Continuous functions

For any two CPO's we denote by $[D \to E]$ the set of continuous functions that map D to E. Considering the pointwise ordering on two functions $f, g \in [D \to E]$, $f \sqsubseteq g$ iff $\forall d \in D. f(d) \sqsubseteq g(d)$, then is $([D \to E], \sqsubseteq)$ a CPO?

This is equivalent with given a chain $f_1, f_2, ...$ with $f_i \in [D \to E]$ the LUB of the chain is also in $[D \to E]$.

We know that we can get a LUB for $D\&E$ we only have to prove that the LUB of $D \to E$ is a continuous function which means that the LUB of a chain of functions is continuous.

The LUB of the chain is also a function so in order to be continuous it must fulfill the equation:

$$(\lambda d \in D. \bigsqcup_{n\in\omega} f_n(d))(\bigsqcup_{m\in\omega} d_m) = \bigsqcup_{m\in\omega} (\lambda d \in D. \bigsqcup_{n\in\omega} f_n(d))(d_m)$$

If we apply $\bigsqcup_{n\in\omega} f_n$ to a chain $d_m \in D$ the LUB of the chain obtained is:

$$\bigsqcup_{m\in\omega} (\lambda d \in D. \bigsqcup_{n\in\omega} f_{n\in D}(dn) = \bigsqcup_{n\in\omega} \bigsqcup_{n\in\omega} f_n(d_m)$$

If we apply $\bigsqcup_{n\in\omega} f_n$ to the $\bigsqcup_{m\in\omega} d_m$ then we get

$$\bigsqcup_{n\in\omega} f_n(\bigsqcup_{m\in\omega} d_m) = \bigsqcup_{n\in\omega} \bigsqcup_{m\in\omega} f_n(d_m)$$

We need to prove now that:

$$\bigsqcup_{n\in\omega} \bigsqcup_{m\in\omega} f_n(d_m) = \bigsqcup_{m\in\omega} \bigsqcup_{n\in\omega} f_n(d_m)$$

We will use the exchange lemma to prove that the order of the joins is commutative:

If f is monotonic and continuous then:

$$\bigsqcup_n \bigsqcup_m f_n(d_m) = \bigsqcup_n f_n(d_n) = \bigsqcup_m \bigsqcup_n f_n(d_m)$$

Proof:

Let $e_{nm} = f_n(d_m)$ then we then notice that

$$n \leq n', m \leq m' \Rightarrow e_{nm} \sqsubseteq e_{n'm'}$$

by choosing $m' = n' = max(m, n)$ we get

$$e_{nm} \sqsubseteq e_{nn}$$

(Suppose WLOG that $m \leq n$) by taking the LUB over n we get

$$\bigsqcup_n e_{nm} \sqsubseteq \bigsqcup_n e_{nn}$$

by taking again the LUB over m then we get

$$\bigsqcup_m \bigsqcup_n e_{nm} \sqsubseteq \bigsqcup_m \bigsqcup_n e_{nn}$$

but e_{nn} is not dependent on m so

$$\bigsqcup_m \bigsqcup_n e_{nm} \sqsubseteq \bigsqcup_n e_{nn}$$

similarly by taking the LUB in the inverse order we get

$$\bigsqcup_n \bigsqcup_m e_{nm} \sqsubseteq \bigsqcup_n e_{nn}$$

Further more if we choose $m' = n' = min(m, n)$ then

$$e_{nn} \sqsubseteq e_{mn}$$

(Suppose WLOG that $n \leq m$) by taking the LUB over n we get

$$\bigsqcup_n e_{nn} \sqsubseteq \bigsqcup_n e_{mn}$$

by taking again the LUB over m then we get

$$\bigsqcup_m \bigsqcup_n e_{nn} \sqsubseteq \bigsqcup_m \bigsqcup_n e_{mn}$$

but e_{nn} is not dependent on m so

$$\bigsqcup_n e_{nn} \sqsubseteq \bigsqcup_m \bigsqcup_n e_{mn}$$

similarly by taking the LUB in the inverse order we get

$$\bigsqcup_n e_{nn} \sqsubseteq \bigsqcup_n \bigsqcup_m e_{mn}$$

so we conclude that

$$\bigsqcup_m \bigsqcup_n e_{nm} = \bigsqcup_n e_n$$

and

$$\bigsqcup_n e_n = \bigsqcup_n \bigsqcup_m e_{nm}$$

which really mean

$$\bigsqcup_m \bigsqcup_n e_{nm} = \bigsqcup_n e_n = \bigsqcup_n \bigsqcup_m e_{nm}$$

and that is

$$\bigsqcup_n \bigsqcup_m f_n(d_m) = \bigsqcup_n f_n(d_n) = \bigsqcup_m \bigsqcup_n f_n(d_m)$$

□

So the LUB of a chain of continuous functions is continuous which means that $[D \leftarrow E]$ is a CPO.

□

1 REC$^+$ Language

Syntactic Forms

$$\begin{array}{rcl} d & ::= & f_1(x_1, \ldots, x_{a_1}) = e_1 \ldots f_n(x_1, \ldots, x_{a_n}) = e_n \\ e & ::= & n \mid X \mid e_1 \oplus e_2 \mid \text{ifp } e_0 \text{ then } e_1 \text{ else } e_2 \mid \text{let } x = e_1 \text{ in } e_2 \mid f_1(e_1, e_2, \ldots, e_{a_1}) \end{array}$$

2 Evaluation Contexts

$$E \;:=\; [\cdot] \;|\; [\cdot] \oplus e \;|\; n \oplus [\cdot] \;|\; [\cdot] \wedge e \;|\; \textsf{ifp } [\cdot] \textsf{ then } e_1 \; else \; e_2 \;|\; \textsf{let } x = [\cdot] \textsf{ in } e$$
$$|\; f_1(v_1, v_2, \ldots, [\cdot], e_k, \ldots, e_{a_i})$$

3 Operational Semantics

Values may be defined as:

$$v \;::=\; n$$

Our evaluation relation is the following:

$e \xrightarrow{d} e'$

The actual semantics are:

$$\frac{n_3 = n_1 + n_2}{n_1 \oplus n_2 \xrightarrow{d} n_3} \qquad \frac{}{0 \wedge e \xrightarrow{d} 0} \qquad \frac{n \neq 0}{n \wedge e \xrightarrow{d} e}$$

$$\frac{n > 0}{\textsf{ifp } n \textsf{ then } e_1 \textsf{ else } e_2 \xrightarrow{d} e_1} \qquad \frac{n \leq 0}{\textsf{ifp } n \textsf{ then } e_1 \textsf{ else } e_2 \xrightarrow{d} e_2} \qquad \frac{}{\textsf{let } x = n \textsf{ in } e \xrightarrow{d} e\{n/x\}}$$

For a CBN evaluation style, we would include the following:

$$\frac{f_i(x_i, \ldots, x_{a_i}) = e_i \in d}{f_i(n_1, \ldots, n_{a_i}) \xrightarrow{d} e_i\{n_1/x_1, \ldots, n_{a_i}/x_{a_i}\}}$$

For a CBV evaluation style, we would include the following instead:

$$\frac{f_i(x_i, \ldots, x_{a_i}) = e_i \in d}{f_i(e'_1, \ldots, e'_{a_i}) \xrightarrow{d} e_i\{e'_1/x_1, \ldots, e'_{a_i}/x_{a_i}\}}$$

An example program in REC$^+$: (this program finds the first prime after 1000)

$f_1(n, m) = n - m * n \wedge (n = m * (n/m) \vee f_1(n, m + 1))$
$f_2(n) = f_1(n, 2)$
$f_3(n) = \textsf{ifp } f_2(n) \textsf{ then } n \textsf{ else } f_2(n + 1) \textsf{ in}$
$f_3(1000)$

4 Denotational Semantics

What is our semantic function? $\mathcal{E}[\![e]\!] : \mathbb{Z}$? Possibly, but it depends on the function declarations. So we need an environment:

$$\phi \in \mathit{FEnv} \;=\; (\mathbb{Z}^{a_1} \longrightarrow \mathit{Result}) \times (\mathbb{Z}^{a_2} \longrightarrow \mathit{Result}) \times \;\ldots\; \times (\mathbb{Z}^{a_n} \longrightarrow \mathit{Result})$$

$$
\begin{aligned}
& \langle F_1, \ldots, F_n \rangle \\
\rho \in \mathit{Env} \quad = \quad & \mathsf{Var} \longrightarrow \mathit{Result}
\end{aligned}
$$

Where $\mathit{Result} = \mathbb{Z}_\perp$. So, our semantic functions look like:

$$\mathcal{E} \quad : \quad \mathit{Exp} \longrightarrow \mathit{FEnv} \longrightarrow \mathit{Env} \longrightarrow \mathit{Result}$$

$$
\begin{aligned}
\mathcal{E}[\![n]\!]\phi\rho &= \lfloor n \rfloor \\
\mathcal{E}[\![x]\!]\phi\rho &= \lfloor \rho(x) \rfloor \\
\mathcal{E}[\![e_1 \oplus e_2]\!]\phi\rho &= \mathsf{let}\ v_1 \in \mathbb{Z} = \mathcal{E}[\![e_1]\!]\phi\rho\ \mathsf{in} \\
&\quad\ \mathsf{let}\ v_2 \in \mathbb{Z} = \mathcal{E}[\![e_2]\!]\phi\rho\ \mathsf{in} \\
&\quad\quad \lfloor v_1 \oplus v_2 \rfloor
\end{aligned}
$$

Alternately, we could lift the $\oplus$ operation:

$$\mathcal{E}[\![e_1 \oplus e_2]\!]\phi\rho \quad = \quad \mathcal{E}[\![e_1]\!]\phi\rho \ \oplus_\perp \ \mathcal{E}[\![e_2]\!]\phi\rho$$

Continuing with our semantics:

$$
\begin{aligned}
\mathcal{E}[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!]\phi\rho &= \mathsf{let}\ y \in \mathbb{Z} = \mathcal{E}[\![e_1]\!]\phi\rho\ \mathsf{in} \\
&\quad\ \mathcal{E}[\![e_2]\!]\phi\rho\ [x \mapsto y] \\
\mathcal{E}[\![f_i(e_1, \ldots, e_{a_i})]\!]\phi\rho &= \mathsf{let}\ v_1 \in \mathbb{Z} = \mathcal{E}[\![e_1]\!]\phi\rho\ , \\
&\quad\quad \vdots \\
&\quad\quad v_{a_i} \in \mathbb{Z} = \mathcal{E}[\![e_{a_i}]\!]\phi\rho\ \mathsf{in} \\
&\quad\ (\pi_i\phi)\langle v_1, \ldots, v_{a_i} \rangle
\end{aligned}
$$

What if we wanted to write a CBN environment instead of a CBV environment? Our domain equations would change:

$$
\begin{aligned}
\mathit{Env} &= \mathsf{Var} \longrightarrow \mathbb{Z}_\perp \\
\mathit{FEnv} &= (\mathbb{Z}_\perp^{a_1} \longrightarrow \mathbb{Z}_\perp) \ \times \ (\mathbb{Z}_\perp^{a_2} \longrightarrow \mathbb{Z}_\perp) \ \times \ \ldots \ \times \ (\mathbb{Z}_\perp^{a_n} \longrightarrow \mathbb{Z}_\perp)
\end{aligned}
$$

As would our semantic functions:

$$
\begin{aligned}
\mathcal{E}[\![x]\!]\phi\rho &= \rho(x) \\
\mathcal{E}[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!]\phi\rho &= \mathcal{E}[\![e_2]\!]\phi\rho\ [x \mapsto \mathcal{E}[\![e_1]\!]\phi\rho] \\
\mathcal{E}[\![f_i(e_1, \ldots, e_{a_i})]\!]\phi\rho &= (\pi_i\phi)\langle \mathcal{E}[\![e_1]\!]\phi\rho, \ldots, \mathcal{E}[\![e_{a_i}]\!]\phi\rho \rangle
\end{aligned}
$$

There's only one issue left with our denotational semantics: where does ϕ come from? Our meaning function:

$$
\begin{aligned}
& \mathcal{D}[\![d]\!] \ \in \ \mathit{FEnv} \\
& \mathcal{E}[\![e]\!]\mathcal{D}[\![d]\!](\lambda x.\phi) \quad = \quad [\![d\ \mathsf{in}\ e]\!]
\end{aligned}
$$

We'll pick up with the definition of $\mathcal{D}$ in the next lecture.