CS611 Lecture 12 **Domain Constructions II** 24 September 2003

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1 Review

Recall that we want to define the denotational semantics of the while loop as:

$$\begin{aligned}\mathcal{C}[\![\textbf{while}\, b\, \textbf{do}\, c]\!] &= \mathit{fix}(\Gamma)\\ &= \bigsqcup_{n\in\mathbb{N}} \Gamma^n(\perp_{\Sigma\to\Sigma_\perp})\end{aligned}$$

For this to be a well-formed definition, we must show that that Γ is continuous. But Γ is a complex function:

$$\begin{aligned}\Gamma(\gamma) = \lambda\sigma.\quad & \text{if } (\neg\mathcal{B}[\![b]\!]\sigma) \text{ then } \sigma\\ & \text{else if } (\mathcal{C}[\![c]\!]\sigma = \perp) \text{ then } \perp\\ & \text{else } \gamma(\mathcal{C}[\![c]\!]\sigma)\end{aligned}$$

It is non-trivial to prove that this is a continuous function. The goal of Domain Constructions is to provide us techniques to prove the continuity of such complex functions.

2 Domain Constructions of functions

2.1 Composition

Claim: $f : D \to E, g : C \to D$ are both continuous functions, then we have $f \circ g : C \to E$ is also continuous.

To prove this, we should prove two things:

1. This function $f \circ g : C \to E$ is monotonic
2. This function $f \circ g : C \to E$ for any ω-chain $\{(x_n)\}_{n\in\mathbb{N}}$ in C,

$$(f\circ g)(\bigsqcup_{i\geq 0} x_i) = \bigsqcup_{i\geq 0}(f\circ g)(x_i)$$

Proof:

1. to show that $f \circ g$is monotonic, let $x \sqsubseteq y$, then

$$\begin{aligned} & g(x) \sqsubseteq g(y) && \text{(by g is monotonic)}\\ \Rightarrow\ & f(g(x)) \sqsubseteq f(g(y)) && \text{(by f is monotonic)}\\ \Rightarrow\ & (f\circ g)(x) \sqsubseteq (f\circ g)(y) && \text{(by definition of } \circ)\end{aligned}$$

2. to show that $f \circ g$ is continuous, Take $\{x_i\}_{i\in\mathbb{N}}$, ω-chain in C, we want

$$(f\circ g)(\bigsqcup_{i\geq 0} x_i) = \bigsqcup_{i\geq 0}(f\circ g)(x_i)$$

$$\begin{aligned} f(g(\textstyle\bigsqcup_{i\geq 0} x_i) &= f(\textstyle\bigsqcup_{i\geq 0} g(x_i)) && \text{(by g is continuous)}\\ &= \textstyle\bigsqcup_{i\geq 0} f(g(x_i)) && \text{(by f is continuous)}\\ &= \textstyle\bigsqcup_{i\geq 0} (f\circ g)(x_i) && \text{(by definition of } \circ)\end{aligned}$$

2.2 Product of Functions

Claim: Take k continuous functions, $f_1 : E_1 \to D_1, \cdots, f_k : E_k \to D_k$
Then the function

$$(f_1 \times f_2 \times \cdots \times f_k) : (E_1 \times E_2 \times \cdots \times E_k) \to (D_1 \times D_2 \times \cdots \times D_k)$$

$$(f_1 \times f_2 \times \cdots \times f_k)(e_1, e_2, \cdots, e_k) = (f_1(e_1), \cdots, f_k(e_k))$$

is continuous.
Proof:
We can see that the product can be defined as

$$f_1 \times f_2 \times \cdots \times f_k = \langle f_1 \circ \Pi_1, f_2 \circ \Pi_2, \cdots, f_k \circ \Pi_k \rangle$$

by continuous function's projection, composition, and tupling are continuous, so their product is continuous.

2.3 Let

$$f : E \rightarrow (D_1 \times D_2 \times \cdots \times D_k)$$

Then f is continuous $\Leftrightarrow \forall i \leq k.(\Pi_i \circ f)$ is continuous
Proof:
$\Rightarrow$: Trivial, by continuity of composition and projection.
$\Leftarrow$: $f = \langle \Pi_1 \circ f, \cdots, \Pi_k \circ f \rangle$ by continuity of tupling

2.4 Let

$$f : (E_1 \times E_2 \times \cdots \times E_k) \rightarrow D$$

f is continuous $\Leftrightarrow \forall i, \forall e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k.\ f_i : E_i \rightarrow D$ defined as $f_i(e_i) = f(e_1, \cdots, e_i, \cdots, e_k)$ is continuous.
Proof:
$\Rightarrow$: We can write f_i as $f_i = f \circ g_i$ where g_i is defined as $g_i : E_i \rightarrow (E_1 \times \cdots \times E_k)$
For example: $g_i(e_i) = (e_1, \cdots, e_i, \cdots, e_k)$.
Under this definition, $\forall j.\Pi_j \circ g_i$ is the identity function for i=j and a constant function if $i \neq j$
Now we show that $\forall j \leq k.\ \Pi_j \circ g_i$ is continuous. This is true because:

1. for $i \neq j$ it is a constant function, so in this case $\Pi_j \circ g_i$ is continuous
2. for $i = j$ it is the identity, so in this case $\Pi_j \circ g_i$ is continuous

so in any case $\Pi_j \circ g_i$ is continuous.
$\Rightarrow$ by previous claim(2.3) we get g_i is continuous
$\Rightarrow$by composition $f \circ g_i$ is continuous
$\Rightarrow f_i$ is continuous

$\Leftarrow$: Assume $k = 2$ (to make the proof succinct)
Assume f is continuous in its first and second argument. Then:

$$f(\bigsqcup_{n \geq 0} (x_n, y_n)) = f(\bigsqcup_{m \geq 0} x_m, \bigsqcup_{n \geq 0} y_n) \quad \text{(l.u.b. of pairs)}$$

$$= \bigsqcup_{m \geq 0} f(x_m, \bigsqcup_{n \geq 0} y_n) \quad \text{(continuity on the first argument)}$$

$$= \bigsqcup_{m \geq 0} \bigsqcup_{n \geq 0} f(x_m, y_n) \quad \text{(continuity on the second argument)}$$

What is left to prove is:

$$\bigsqcup_{m \geq 0} \bigsqcup_{n \geq 0} f(x_m, y_n) = \bigsqcup_{n \geq 0} f(x_n, y_n)$$

This is non-trivial, and is known as the Exchange Lemma.

2.5 Exchange Lemma
Let $\{x_{m,n} | m \in \mathbb{N}, n \in \mathbb{N}\}$, elements of a cpo $(D, \sqsubseteq)$ such that $\forall m \leq m', n \leq n'.\ x_{mn} \sqsubseteq x_{m'n'}$
Then

$$\bigsqcup_{m \geq 0} \bigsqcup_{n \geq 0} x_{mn} = \bigsqcup_{n \geq 0} \bigsqcup_{m \geq 0} x_{mn} = \bigsqcup_{n \geq 0} x_{nn}$$

Proof:
We will show that

1. $$\bigsqcup_m \bigsqcup_n x_{mn} \sqsubseteq \bigsqcup_n x_{nn}$$

   Let$x = \bigsqcup_{n'} x_{n'n'}$ from definition of l.u.b we have $\forall n'.x_{n'n'} \sqsubseteq x$
   Take arbitrary m,n and consider x_{mn}, take $p = \max(m,n) \Rightarrow x_{mn} \sqsubseteq x_{pp} \sqsubseteq x$
   $\Rightarrow \forall m,n.x_{mn} \sqsubseteq x \ \Rightarrow \forall m. \bigsqcup_n x_{mn} \sqsubseteq x \ \Rightarrow \bigsqcup_m \bigsqcup_n x_{mn} \sqsubseteq x.$

2. $$\bigsqcup_n x_{nn} \sqsubseteq \bigsqcup_n \bigsqcup_m x_{nm}$$

   $$\begin{array}{lll} y = \bigsqcup_m \bigsqcup_n x_{mn} & \Rightarrow \forall m \ . \ \bigsqcup_n x_{mn} \sqsubseteq y & \text{(by definition of } \bigsqcup_m) \\ & \Rightarrow \forall m,n \ . \ x_{mn} \sqsubseteq y & \text{(by definition of } \bigsqcup_n) \\ & \Rightarrow \forall n \ . \ x_{nn} \sqsubseteq y & \text{(in particular when m=n)} \\ & \Rightarrow \bigsqcup_n x_{nn} \sqsubseteq y & \text{(by definition of } \bigsqcup_n) \end{array}$$

Hence $\bigsqcup_m \bigsqcup_n x_{mn} = \bigsqcup_n x_{nn}$. The proof for $\bigsqcup_n \bigsqcup_m x_{mn} = \bigsqcup_n x_{nn}$ is similar.

2.5 Function Spaces

We already show that if $(B, \sqsubseteq)$ is a cpo then $(A \to B, \sqsubseteq)$ is a cpo.
Now let $[A \to B]$ be $\{f \in A \to B | f \text{ is continuous}\}$
if $(A, \sqsubseteq), (B, \sqsubseteq)$ are cpo's, we want to show that $([A \to B], \sqsubseteq)$ is also a cpo.
We know $([A \to B], \sqsubseteq)$ is a partial order, now to see if it is a c.p.o.
We want to check if $\forall \omega - chain.\{f_n\}_{n \geq 0}. \bigsqcup_{n \geq 0} f_n$ is also a continuous function.
this is, $\bigsqcup_{n \geq 0} f_n \in [A \to B]$

$$\begin{array}{lll} (\bigsqcup_{n \geq 0} f_n)(\bigsqcup_{m \geq 0} d_m) & = \bigsqcup_{n \geq 0} (f_n(\bigsqcup_{m \geq 0} d_m)) & \text{(by definition of l.u.b for functions)} \\ & = \bigsqcup_{n \geq 0} \bigsqcup_{m \geq 0} (f_n(d_m)) & \text{(all } f_n \text{ are continuous)} \\ & = \bigsqcup_{m \geq 0} \bigsqcup_{n \geq 0} (f_n(d_m)) & \text{(Exchange Lemma)} \\ & = \bigsqcup_{m \geq 0} (\bigsqcup_{n \geq 0} (f_n(d_m))) & \text{(by definition of l.u.b for functions)} \\ & = \bigsqcup_m ((\bigsqcup_n f_n)(d_m)) & \end{array}$$