CS611 Lecture 11 **Domain Constructions I** 22 September 2003

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1 Denotational Semantics of "While Loop"

In lecture 9, we defined the denotational semantics of **while** b **do** c to be the least fixed point of a higher order function Γ:

$$\mathcal{C}[\![\textbf{while}\, b\, \textbf{do}\, b]\!] = \text{fix}(\Gamma)$$

where:

$$\begin{aligned}\Gamma(\gamma) = \lambda\sigma.\ \ & \text{if } (\neg\mathcal{B}[\![b]\!]\sigma) \text{ then } \sigma \\ & \text{else if } (\mathcal{C}[\![c]\!]\sigma = \perp) \text{ then } \perp \\ & \text{else } \gamma(\mathcal{C}[\![c]\!]\sigma)\end{aligned}$$

where $\gamma : \Sigma \to \Sigma_\perp$. More precisely, we claimed that the least fixed point of Γ (and thus the denotation of the while loop) is:

$$\mathcal{C}[\![\textbf{while}\, b\, \textbf{do}\, c]\!] = \bigsqcup_{n\in\mathbb{N}} \Gamma^n(\perp_{\Sigma\to\Sigma_\perp})$$

where the least upper bound is taken with respect to the pointwise ordering over the function domain $\Sigma \to \Sigma_\perp$. This is clearly a cpo (and therefore the l.u.b. in the above formula is well defined) because of the following:

- The domain of states $(\Sigma, \sqsubseteq_2)$ is a cpo with bottom, where the ordering of states $\sqsubseteq_2$ is:

  $$\sigma \sqsubseteq_2 \sigma' \ \text{ iff } \ \sigma = \sigma' \ \vee \ \sigma = \perp$$

  To prove this, one must show that:

  1. The relation $\sqsubseteq_2$ is a partial order;
  2. The relation $\sqsubseteq_2$ is a complete partial order, by showing that any ω-chain has an l.u.b.. This is easy, because every ω-chain can have at most two distinct states, $\perp$ and σ. Hence, its lub is either $\perp$ or σ.
  3. $\perp$ is the bottom of this cpo.

- The function domain $\Sigma \to \Sigma_\perp$, with the pointwise ordering, is a cpo with bottom. This domain is a cpo because its range is a cpo (see last lecture). Futhermore, one can show that $\perp_{\Sigma\to\Sigma_\perp} = \lambda\sigma\ .\ \perp$ is the bottom element of this function domain.

Therefore, the expression $\bigsqcup_{n\in\mathbb{N}} \Gamma^n(\perp_{\Sigma\to\Sigma_\perp})$ is well-defined. But why is it the least fixed point of Γ? For what class of functions does this formula yield the least fixed point? We will first give some definitions, then state and prove the Least Fixed Point Theorem, which gives the answers to these questions.

2 Properties of functions

2.1 Monotonicity

First let's introduce the definition of *monotonicity*.

Definition (Monotonicity)
Let $(E, \sqsubseteq_E), (D, \sqsubseteq_D)$ be two posets. Then a function $f : E \to D$ is monotonic iff

$$\forall x, y \in E. x \sqsubseteq_E y \Rightarrow f(x) \sqsubseteq_D f(y)$$

.

Now let's ask, what properties should a function f have in order to show that $\bigsqcup_{n\in\mathbb{N}} f^n(\perp) = \text{fix}(f)$?

- Do we need monotonicity? Yes, because we want $\{f^n\}_{n\in\mathbb{N}}$ to be an ω-chain. Otherwise, the l.u.b. of this sequence is not defined and we can't talk about $\bigsqcup_{n\in\mathbb{N}} f^n(\perp)$.
- Is monotonicity enough? No. For this we'll show a counter example, that is a monotonic function f, for which $\bigsqcup_{n\in\mathbb{N}} f^n(\perp) \neq \text{fix}(f)$.

  Example: $([0,1],\leq)$ is a cpo, where $\perp \equiv 0$. Suppose

  $$f(x) = \begin{cases} \frac{x}{2} + \frac{1}{4} & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{otherwise} \end{cases}$$

  $f(x)$ is monotonous. consider $\{f^n(0)\}_{n\in\mathbb{N}}$,

  $$\perp \sqsubseteq f(0) \sqsubseteq f^2(0) \sqsubseteq \cdots$$

  that is,

  $$0 \leq \frac{1}{4} \leq \frac{1}{4} + \frac{1}{8} \leq \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \leq \cdots$$

  We can see that

  $$\lim_{n\to\infty} f^n(0) = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}$$

  Therefore, $\bigsqcup_{n\in\mathbb{N}} f^n(0) = \frac{1}{2}$. But $\frac{1}{2}$ is obviously not a fixed point, because $f(\frac{1}{2}) = 1$. Actually, the only fixed point (least fixed point) is 1.

Hence monotonicity is not enough. The key property missing here is *continuity.*

2.2 Continuous Function

Definition (Continuous Function)
Let $(E, \sqsubseteq_E), (D, \sqsubseteq_D)$ be two cpo's. Then a function $f : E \to D$ is continuous iff

1. *it is monotonic*
2. *if for any ω-chain $\{x_i\}_{i\in\mathbb{N}}$ in E,*

   $$f(\bigsqcup_{n\in\mathbb{N}} x_n) = \bigsqcup_{n\in\mathbb{N}} f(x_n)$$

3 Fixed Point Theorem

If $(D, \sqsubseteq, \perp)$ is a cpo with bottom, and $f : D \to D$ is continuous, then:

$$x = \bigsqcup_{n\in\mathbb{N}} f^n(\perp) \textit{ is the least fixed point of } f.$$

Proof:

- x is a fixed point

  $$\begin{aligned} f(x) &= f(\textstyle\bigsqcup_{n\in\mathbb{N}} f^n(\perp)) = \bigsqcup_{n\in\mathbb{N}} f(f^n(\perp)) = \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\perp) = \bigsqcup_{n\in\mathbb{N}}\{f^n(\perp) | n \geq 1\} \\ &= \textstyle\bigsqcup_{n\in\mathbb{N}}\{\perp\} \cup \{f^n(\perp) | n \geq 1\} \text{(for any } \omega\text{-chain, adding a } \perp \text{ to it won't change its l.u.b.)} \\ &= \textstyle\bigsqcup_{n\in\mathbb{N}} f^n(\perp) = x \end{aligned}$$

- x is a least fixed point

  Let y be another fixed point (i.e., $y = f(y)$), we want $x \sqsubseteq y$.

  Claim: $\forall n.\ f^n(\perp) \sqsubseteq y$

  We will prove this by standard induction on n.

  - Base case: $n = 0$. $\perp \sqsubseteq y$ holds.
  - Inductive Step: Assume $f^n(\perp) \sqsubseteq y$, then by monotonicity of f,
    $f(f^n(\perp)) \sqsubseteq f(y) \Rightarrow f^{n+1}(\perp) \sqsubseteq f(y) = y$

  Hence, we get $\forall n.\ f^n(\perp) \sqsubseteq y$.

  By the definition of least upper bound, we have $\bigsqcup_{n\in\mathbb{N}} f^n(\perp) \sqsubseteq y$

Now we want to prove that Γ is continuous, because if it is, and provided we have already proved $(\Sigma \to \Sigma_\perp, \sqsubseteq_1)$ is a cpo with bottom, the *fixed point theorem* tells us that $\bigsqcup_{n\in\mathbb{N}} \Gamma^n(\perp_{\Sigma\to\Sigma_\perp})$ is the least fixed point of Γ. But Γ is a complex function and it is not trivial to show that it is continuous. We need a theory that allows us to prove that complex functions, such as Γ, are continuous.

4 Domain Constructions

Domain constructions will allow us to systematically create new cpo's from existing cpo's, and to create new continuous functions from existing continuous functions.

- Product domains

  $(D_1, \sqsubseteq_1), (D_2, \sqsubseteq_2)$ are cpo's. Then define $\sqsubseteq$ over $D_1 \times D_2$

  $$(a, b) \sqsubseteq (c, d) \text{ iff } (a \sqsubseteq_1 c) \wedge (b, \sqsubseteq_2 d)$$

  Then $(D_1 \times D_2, \sqsubseteq)$ is a cpo.

  (consider any ω-chain $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ in $D_1 \times D_2$, $\bigsqcup_{n\in\mathbb{N}}(a_n, b_n) = (\bigsqcup_{n\in\mathbb{N}} a_n, \bigsqcup_{n\in\mathbb{N}} b_n))$ (1)

  To generalize: $(D_1 \times \cdots \times D_k, \sqsubseteq)$ is also a cpo.

- Tupling

  Take k continuous functions, $f_1 : E \to D_1, \cdots, f_k : E \to D_k$, where $E, D_1, \cdots, D_k$ are cpo's.

  Define $\langle f_1, \cdots, f_k\rangle : E \to D_1 \times \cdots \times D_k$ as:

  $$\langle f_1, \cdots, f_k\rangle(e) = (f_1(e), \cdots, f_k(e))$$

  We want to prove $\langle f_1, \cdots, f_k\rangle$ is continuous.

  - Monotonicity

  $$\begin{aligned} e \sqsubseteq e' \quad &\Rightarrow \forall i.\ f_i(e) \sqsubseteq f_i(e') \\ &\Rightarrow (f_1(e), \cdots, f_k(e)) \sqsubseteq (f_1(e'), \cdots, f_k(e')) \\ &\Rightarrow \langle f_1, \cdots, f_k\rangle(e) \sqsubseteq \langle f_1, \cdots, f_n\rangle(e') \end{aligned}$$

  - Continuity

  $$\begin{aligned} &\langle f_1, \cdots, f_k\rangle(\textstyle\bigsqcup_{n\in\mathbb{N}}(x_n)) \\ &\quad = (f_1(\textstyle\bigsqcup_{n\in\mathbb{N}} x_n), \cdots, f_k(\bigsqcup_{n\in\mathbb{N}}(x_n)) \text{ (by definition of } \langle f_1, \cdots, f_k\rangle) \\ &\quad = (\textstyle\bigsqcup_{n\in\mathbb{N}} f_1(x_n), \cdots, \bigsqcup_{n\in\mathbb{N}} f_k(x_n)) \text{ (by continuity of } f_1, \cdots, f_k) \\ &\quad = \textstyle\bigsqcup_{n\in\mathbb{N}}(f_1(x_n), \cdots, f_k(x_n))\text{(by applying the generalized version of (1))} \\ &\quad = \textstyle\bigsqcup_{n\in\mathbb{N}}\langle f_1, \cdots, f_k\rangle(x_n) \text{ (by definition of } \langle f_1, \cdots, f_k\rangle) \end{aligned}$$

- Projection

  Let $D_1, \cdots, D_k$ be cpo's, and $\Pi_i : D_1 \times \cdots \times D_k \to D_i$

$$\Pi_i(d_1, \cdots, d_k) = d_i$$

  One can show that Π_i is continuous (proof left as an exercise).