CS611 Lecture 06 **Complete Partial Orders** 19 September 2003

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1 Partial Orders

A partially ordered set (poset) is a pair $(S, \sqsubseteq)$ consisting of a set S, and a binary relation $\sqsubseteq$ on that set, such that $\sqsubseteq$ is:

- reflexive: $\forall x \in S \,.\, x \sqsubseteq x$
- antisymmetric: $\forall x, y \in S \,.\, (x \sqsubseteq y \wedge y \sqsubseteq x) \Rightarrow x = y$
- transitive: $\forall x, y, z \in S \,.\, (x \sqsubseteq y \wedge y \sqsubseteq z) \Rightarrow x \sqsubseteq z$

Typical examples of posets:

- $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$ are posets
- $(\mathbb{N}, =)$ is a poset
- If $(A, \sqsubseteq_A)$, $(B, \sqsubseteq_B)$ are posets, and we define $\sqsubseteq$ over the domain $A \times B$,

$$(a, b) \sqsubseteq (a', b') \textbf{ iff } a \sqsubseteq_A a', b \sqsubseteq_B b'$$

  then $(A \times B, \sqsubseteq)$ is poset. (Such a relation is called component-wise ordering.)

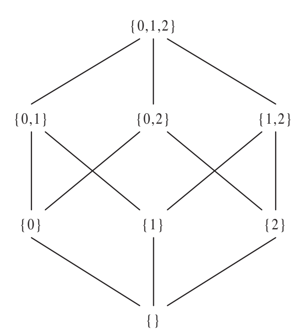
Non-examples of posets:

- $(\mathbb{N}, <)$ is not poset. (Because it is not reflexive)
- $(\mathbb{Z}, |)$ is not poset. (Because it is not antisymmetric. e.g. $2|-2$ & $-2|2$)

2 Hasse Diagrams

Posets can be represented by Hasse diagrams, which have the following characteristics:

- Each element in the poset is a node in the diagram.
- There is an edge (x, y) **iff** $x \sqsubseteq y$, and there is no z such that $x \sqsubseteq z$ and $z \sqsubseteq y$.
- If $x \sqsubseteq y$, then x appears below y in the diagram and there should be a path from x to y in the graph.
- Example: $(\mathcal{P}(\{0, 1, 2\}), \subseteq)$

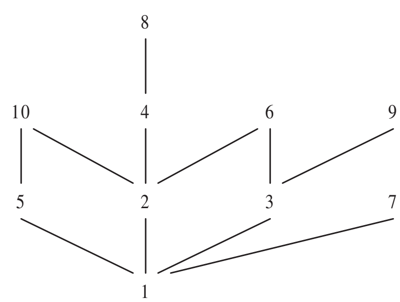


- Another example: $(\mathbb{N}, \leq)$

- The discrete order: $(\mathbb{N}, =)$

  There are no edges between the elements because there are no relations between the elements.

- $(\{1, \ldots, 10\}, |)$



3 Complete Partial Orders

3.1 Upper Bounds

Consider a poset $(S, \sqsubseteq)$ and a set $X \subseteq S$. Then q is an upper bound for X if $\forall p \in X \,.\, p \sqsubseteq q$

3.2 Least Upper Bounds

We say that q is the least upper bound for X, if

- it is an upper bound for X
- $\forall q' \,.\, (\forall p \in X \,.\, p \sqsubseteq q') \Rightarrow q \sqsubseteq q'$

We use the following notation for least upper bounds: $\bigsqcup X$. Least upper bounds are also referred to as lub or L.U.B.

3.3 Chains

A chain is a totally ordered subset of a poset.

$$\ldots \sqsubseteq x_{i-1} \sqsubseteq x_i \sqsubseteq \ldots$$

If S is finite, then the chain stabilizes in both directions:

$$\exists n, m \ . \ \begin{cases} x_i &= x_m, \quad \forall i \geq m \\ x_i &= x_n, \quad \forall i \leq n \end{cases}$$

If S is finite then $\forall x \subseteq S$ there exists a lub for x.

3.4 ω-chains

An ω-chain is an increasing sequence

$$x_0 \sqsubseteq x_1 \sqsubseteq \ldots$$

We only deal with ω-chains in denotational semantics.

3.5 Complete Partial Orders

A complete partial order (cpo) is a pair $(S, \sqsubseteq)$ such that

- $(S, \sqsubseteq)$ is a poset
- any ω-chain has a lub

Note: the lub has to be in set S, but it does not have to be in the ω-chain.

3.6 CPO with bottom

$(S, \sqsubseteq, \perp)$ is a cpo with bottom, if there exists an element $\perp$ such that:

$$\perp \in S \text{ and } \forall x \in S \, . \perp \sqsubseteq x$$

4 Lattices

We briefly mention how complete partial orders compare with lattices, which are algebraic structures used in dataflow analysis frameworks. Recall that a pair $(S, \sqsubseteq)$ is a lattice if:

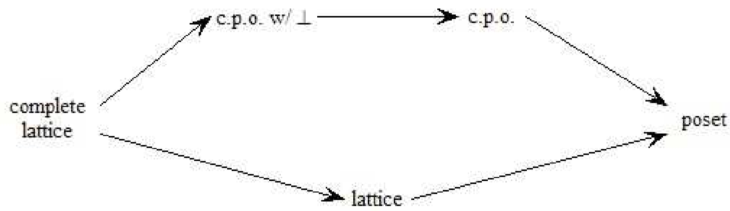
- $(S, \sqsubseteq)$ is a poset
- any finite subset of S has a lub and a glb[1]

$(S, \sqsubseteq)$ is a complete lattice if

- $(S, \sqsubseteq)$ is a poset
- any subset of S has a lub and a glb

The distinction between cpo's and lattices is that in cpo's, we talk about ω-chains, whereas in lattices we speak of arbitrary finite/infinite sets. The diagram below shows the relation between cpo's and lattices, where the arrows indicate implication. Note there is no relation between cpo's and lattices.

[1]glb is the greatest lower bound. It is defined similarly to lub.



5 Examples of cpo's

- $(\mathcal{P}(S), \subseteq)$ is a cpo, $\sqcup \equiv \bigcup$
- $(\mathbb{N}, \leq)$ is not a cpo, because the whole set $\mathbb{N}$ doesn't have a least upper bound. However, $(\mathbb{N} \cup \{\infty\}, \leq)$ is a cpo, where the ordering is extended with the relations $n \leq \infty$ for all n.
- $([0,1], \leq)$ is a cpo, but $([0,1), \leq)$ is not a cpo (counter example: $x_n = 1 - 1/(n+1)$, which has the limit of 1, but $1 \notin [0,1)$)
- $(S, \sqsubseteq)$ is a cpo. Is $(S, \sqsupseteq)$ a cpo?

  No. In a cpo, the reversed chains do not necessarily have a lub.
- Functions:

  Assume $(B, \sqsubseteq_B)$ a cpo and define $\sqsubseteq$ over $A \to B$:

  $$f \sqsubseteq g \textbf{ iff } \forall x \in A \,.\, f(x) \sqsubseteq_B g(x)$$

  which is called a pointwise ordering.

  Then $(A \to B, \sqsubseteq)$ is a cpo. We need to show that: a) it is a partial order; and b) any ω-chain has a least upper bound. We already know the first fact.

  We need to prove that any ω-chain in $A \to B$ has a lub. Consider an ω-chain:

  $$f_0 \sqsubseteq f_1 \sqsubseteq \ldots$$

  Consider a function f such that

  $$f(x) = \bigsqcup_{i \in \mathbb{N}} f_i(x)$$

  Note that $\{f_i(x)\}_{i \geq 0}$ is an ω-chain in B, so f is well defined. One can show that this function f is a least upper bound for the chain $\{f_n\}_{n \geq 0}$. We leave this proof as an exercise.

  Note: There are two steps to prove this:

  - prove f is an upper bound
  - prove f is the least upper bound