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1 Domain construction

In the last lecture we showed that continuity of functions is preserved under composition, product, tupling and projection. In this lecture we will show the constructions of more cpos.

1.1 Function spaces

Let $(E, \sqsubseteq)$ be a cpo then $(D \to E, \sqsubseteq)$ is a cpo if ordered according to

$$f_1 \sqsubseteq f_2 \ \text{ iff } \ \forall d \in D. f_1(d) \sqsubseteq f_2(d)$$

Letting $[D \to E] = \{f \mid f : D \to E,\ f \text{ continuous}\}$, then if $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ are both cpo's then so is $([D \to E], \sqsubseteq)$. To see this we must show that for any ω-chain of continuous functions f_i, $\bigsqcup_n f_n$ is also continuous:

$$\begin{aligned}
(\bigsqcup_n f_n)(\bigsqcup_m d_m) &= \bigsqcup_n f_n(\bigsqcup_m d_m) && \text{(definition of LUB of } f_n) \\
&= \bigsqcup_n (\bigsqcup_m f_n(d_m)) && \text{(each } f_n \text{ is continuous)} \\
&= \bigsqcup_m (\bigsqcup_n f_n(d_m)) && \text{(by Exchange Lemma)} \\
&= \bigsqcup_m ((\bigsqcup_n f_n)(d_m)) && \text{(definition of LUB of } f_n)
\end{aligned}$$

1.2 Operations on Function Spaces

We will consider some functions on the space of continuous functions $[D \to E]$.

1.2.1 Apply

Suppose D and E are two cpo's. We define $\mathit{apply} : ([D \to E] \times D) \to E$ by

$$\mathit{apply}(f, d) = f(d) \qquad \text{(for } f \in [D \to E],\ d \in D.)$$

Recall that a function on a product of cpo's is continuous if it is continuous in each of its arguments separately. Proving monotonicity in each argument is straightforward and is left as an exercise. For the continuity condition, we consider an arbitrary ω-chain $f_0 \sqsubseteq f_1 \sqsubseteq \ldots \subseteq [D \to E]$ and $d \in D$ we have

$$\begin{aligned}
\mathit{apply}(\bigsqcup_n f_n, d) &= (\bigsqcup_n f_n)(d) && \text{(definition of } \mathit{apply}) \\
&= \bigsqcup_n f_n(d) && \text{(LUB is pointwise)} \\
&= \bigsqcup_n \mathit{apply}(f_n, d) && \text{(definition of } \mathit{apply})
\end{aligned}$$

Also, for any $f \in [D \to E]$ and ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \subseteq D$ we have

$$\begin{aligned}
apply(f, \bigsqcup_n d_n) &= f(\bigsqcup_n d_n) \qquad \text{(definition of } apply\text{)} \\
&= \bigsqcup_n f(d_n) \qquad (f \text{ is continuous}) \\
&= \bigsqcup_n apply(f, d_n) \qquad \text{(definition of } apply\text{)}
\end{aligned}$$

1.2.2 Curry

Suppose that F is a cpo and that a function $g\colon(D \times E) \to F$ is continuous. We define $curry(g)\colon D \to [E \to F]$ in λ-notation by

$$curry(g) = \lambda d \in D \,.\, (\lambda e \in E \,.\, g(d, e))$$

1.2.3 FIRE DRILL

1.3 Curry cont'd

Since g is continuous and hence continuous in each argument, we have that $(curry(g))(d)$ for any $d \in D$ is continuous. Also $curry(g)$ is continuous. To see this take an ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \subseteq D$ and $e \in E$, then we have

$$\begin{aligned}
curry(g)(\bigsqcup_n d_n)(e) &= g(\bigsqcup_n d_n, e) \qquad \text{(definition of } curry\text{)} \\
&= \bigsqcup_n g(d_n, e) \qquad \text{(g is continuous)} \\
&= \bigsqcup_n (curry(g)(d_n)(e)) \qquad \text{(definition of } curry\text{)} \\
&= (\bigsqcup_n curry(g)(d_n))(e) \qquad \text{(LUB is pointwise)}
\end{aligned}$$

1.3.1 Composition

Suppose that both $f\colon D \to E$ and $g\colon E \to F$ are continuous. We already know that:

$$g \circ f\colon D \to F \text{ is continuous} \tag{1}$$

We can also prove that:

$$\lambda g.\, \lambda f.\, g \circ f \;:\; ([D \to E] \times [E \to F]) \to [D \to F] \text{ is continuous} \tag{2}$$

Note that (1) and (2) are different statements, and require different proofs. The proof for (1) is left as an exercise.

1.4 Lifting

1. Given a cpo $(D, \sqsubseteq)$, we can extend it to a cpo with bottom $(D_\perp, \sqsubseteq)$. The operation is called lifting and the new cpo is defined as follows.

2. The elements in the new cpo are: $D_\perp \equiv \{\lfloor d \rfloor \mid d \in D\} \cup \{\perp\}$ where $\lfloor - \rfloor$ is some one-to-one function from D to $D_\perp$ s.t. $\lfloor d \rfloor \neq \perp$ for all $d \in D$. In particular, if the original cpo had a bottom element $\perp_D$, then $\lfloor \perp_D \rfloor \neq \perp$.

3. The order in the new cpo is:

$$d_1 \sqsubseteq_{D_\perp} d_2 \ \Leftrightarrow \ (\ (\exists d_1', d_2' \in D \ . \ d_1 = \lfloor d_1' \rfloor \wedge d_2 = \lfloor d_2' \rfloor \wedge d_1' \sqsubseteq_D d_2') \ \vee \ d_1 = \perp)$$

4. Continuous functions on D have a natural extension to continuous functions on $D_\perp$.

   (a) Given that $f : D \to E$ is continuous and E is a cpo with bottom then we can define:

$$\begin{array}{rcl} f^* & : & D_\perp \to E \\ f^*(d') & = & \left\{ \begin{array}{ll} f(d) & \text{if } d' = \lfloor d \rfloor \\ f^*(d') = \perp_E & \text{if } d' = \perp \end{array} \right. \end{array}$$

   (b) One can prove that if f is continuous, then:

$$f^* \ : \ D_\perp \to E \ \text{ is continuous} \tag{3}$$

   (c) One can also prove that:

$$\lambda f.\, f^* \ : \ [D \to E] \to [D_\perp \to E] \ \text{ is continuous} \tag{4}$$

   (d) We can also use the lifting construction to define the let construct:

$$\text{let } x = d' \ . \ e \ \equiv \ (\lambda x.e)^*(d')$$

1.5 Disjoint Sums

1. We next define disjoint sum domain constructions. For simplicity, we first consider the disjoint sum of two cpo's. Let $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ be cpo's. The disjoint sum of these cpo's is a new cpo $(D_1 + D_2, \sqsubseteq)$ defined as follows.

2. We define the set $D_1 + D_2$ using two injective functions in_1, in_2 s.t. $\forall x, y \ in_1(x) \neq in_2(y)$. Then:

$$D_1 + D_2 \equiv \{in_1(d_1) \mid d_1 \in D_1\} \cup \{in_2(d_2) \mid d_2 \in D_2\}$$

3. The partial order on the disjoint sum domain is:

$$\begin{array}{rcl} d \sqsubseteq_{D_1+D_2} d' & \Longleftrightarrow & (d = in_1(d_1) \wedge d' = in_1(d_1') \wedge d_1 \sqsubseteq_{D_1} d_1') \ \vee \\ & & (d = in_2(d_2) \wedge d' = in_2(d_2') \wedge d_2 \sqsubseteq_{D_2} d_2') \end{array}$$

4. We can similarly define the disjoint sum for multiple cpo's $(D_1, \sqsubseteq_1), \ldots, (D_k, \sqsubseteq_k)$, using k injective functions $in_1, \ldots, in_k$.

5. We define functions over disjoint sum domains as follows. Given $f_1 : E_1 \to D, \ldots, f_k : E_k \to D$ continuous, we can define their combination:

$$\begin{array}{rcl} [f_1, \ldots, f_k] & : & (E_1 + \ldots + E_k) \to D \\ [f_1, \ldots, f_k](e) & = & f_i(e_i), \ \text{if } e = in_i(e_i) \end{array}$$

   One can prove that $[f_1, \ldots, f_k] : (E_1 + \ldots + E_k) \to D$ is continuous (left as an exercise).

6. Consider the cpo $\mathbb{U} = (\{u\}, =)$, with a single element (called the *unit domain*). We can then define the domain of truth values as the disjoint sum of $\mathbb{U}$ and itself as $\mathbb{T} = \mathbb{U} + \mathbb{U} = (\{\text{true,false}\}, =)$, using the following injection functions:
$$in_1 : \mathbb{U} \to \mathbb{T},\ in_1(u) = \text{true}$$
$$in_2 : \mathbb{U} \to \mathbb{T},\ in_2(u) = \text{false}$$
We can then define the conditional construction on the disjoint sum domain $\mathbb{T}$. If:
$$\begin{aligned} f_1 &: \mathbb{U} \to E \\ f_2 &: \mathbb{U} \to E \end{aligned}$$
then:
$$\begin{aligned} cond &: (\mathbb{T} \times E \times E) \to E \\ cond(t, e_1, e_2) &= [\lambda x_1.e_1, \lambda x_2.e_2](t) = [f_1, f_2](t) \end{aligned}$$
One can prove that if f_1 and f_2 are continuous, then so is *cond* (left as an exercise).