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1 Continuous Functions

Last time, we ended with a definition of continuous functions. We can make this definition a little more general by not requiring the domain and the range to be the same.

Define: If $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ are complete partial orders (cpo's), then $f : D \to E$ is continuous if:

1. f is monotonic
2. $\forall$ ω-chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D: $\bigsqcup_{n\in\mathbb{N}} f(d_n) = f(\bigsqcup_{n\in\mathbb{N}} d_n)$

(In other words, the function f preserves the least upper bound.)

Recall the Fixed Point Theorem from last time:

Fixed Point Theorem :
If $(D, \sqsubseteq)$ is a cpo with $\perp$ and $f : D \to D$ is a continuous function, then $\mathit{fix}(f) = \bigsqcup_{n\in\mathbb{N}} f^n(\perp)$ is:

1. a fixed point for f
2. the least fixed point

Two lectures ago, we defined Γ as an inductively defined set. We can also define Γ as a function.

$$\Gamma : (\Sigma \to \Sigma_\perp) \to (\Sigma \to \Sigma_\perp)$$

$$\begin{aligned}\Gamma(\gamma) = \lambda\sigma.\ &\textbf{if } \neg\mathcal{B}[\![b]\!]\sigma \textbf{ then } \sigma \\ &\textbf{else if } \mathcal{C}[\![c]\!]\sigma = \perp \textbf{ then } \perp \\ &\textbf{else } \gamma(\mathcal{C}[\![c]\!]\sigma)\end{aligned}$$

Then, we can express the denotational semantics of our favorite while loop w as the fixed point of Γ.

$$\mathcal{C}[\![w]\!] = \mathit{fix}(\Gamma)$$

To apply the Fixed Point Theorem, we need to know that Γ is continuous. But Γ is a fairly complex function, for which it isn't trivial to prove continuity. We therefore need a systematic way to derive complex continuous functions from component continuous functions. Since continuous functions are defined over complete partial order domains, we also need a systematic way of constructing complex cpo domains.

2 Domain Constructions

1. Identity Function $f(x) = x$
   $f : D \to D$ is continuous – trivial.
2. If $f : D \to E$ and $g : E \to H$ are continuous, then $g \circ f : D \to H$ is continuous.
   Proof: $g \circ f$ is monotonic - left as an exercise.
   $g \circ f$ is continuous:

$$\begin{aligned}(g \circ f)(\bigsqcup_n d_n) &= g(f(\bigsqcup_n d_n)) && \text{[by the definition of composition]} \\ &= g(\bigsqcup_n f(d_n)) && \text{[because f is continuous]} \\ &= \bigsqcup_n (g(f(d_n)) && \text{[because g is continuous]} \\ &= \bigsqcup_n (g \circ f)(d_n) && \text{[by the definition of composition]}\end{aligned}$$

2.1 Products

If $(A, \sqsubseteq)$ and $(B, \sqsubseteq)$ are cpo's, then $(A \times B, \sqsubseteq)$ is a cpo where $((a,b) \sqsubseteq (a',b') \iff (a \sqsubseteq a') \wedge (b \sqsubseteq b'))$.

Claim: Given an ω-chain $(a_0, b_0) \sqsubseteq (a_1, b_1) \sqsubseteq \ldots$ in $A \times B$, then the least upper bound of this chain is:

$$\bigsqcup_n (a_n, b_n) = (\bigsqcup_n a_n, \bigsqcup_n b_n)$$

Proof: Left as an exercise.

[Note: We can take the LUB of $\bigsqcup_n a_n$ and $\bigsqcup_n b_n$ because $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ form ω-chains]

Generalization: If $(D_1, \sqsubseteq_1), \ldots, (D_k, \sqsubseteq_k)$ are cpo's, then $((D_1 \times \ldots \times D_k), \sqsubseteq)$ is a cpo, where:

$$(d_1, \ldots, d_k) \sqsubseteq (d'_1, \ldots, d'_k) \text{ iff } (d_1 \sqsubseteq_1 d'_1, \ \ldots \ , d_k \sqsubseteq_k d'_k)$$

2.1.1 Functions over Product Domains

1. *Tupling:* If $f_1 : E \to D_1, \ldots, f_k : E \to D_k$ are continuous, then $\langle f_1, \ldots, f_k \rangle : E \to (D_1 \times \ldots \times D_k)$ is continuous, where $\langle f_1, \ldots, f_k \rangle$ is defined as:

   $$\langle f_1, \ldots, f_k \rangle(e) = (f_1(e), \ldots, f_k(e))$$

   [Note: we need the domain and range of continuous functions to be partial orders. Because f_k's are continuous, it means that E, D_1, ..., D_k are all cpo's. Therefore E and $D_1 \times \ldots \times D_k$ are also cpo's, i.e. the domain and range of $\langle f_1, \ldots, f_k \rangle$ are cpo's.]

   Proof :

   Monotonicity :
   In order to show that $\langle f_1, \ldots, f_k \rangle$ is monotonic, we need to prove that $e_1 \sqsubseteq e_2 \Rightarrow \langle f_1, \ldots, f_k \rangle(e_1) \sqsubseteq \langle f1, \ldots, f_k \rangle(e_2)$. Because our component functions $f_1, \ldots, f_k$ are monotonic and $e_1 \sqsubseteq e_2$, we know that $f_1(e_1) \sqsubseteq f_1(e_2), \ldots, f_k(e_1) \sqsubseteq f_k(e_2)$. We can then apply the tupling function to get $(f_1(e_1), \ldots, f_k(e_1)) \sqsubseteq (f_1(e_2), \ldots, f_k(e_2))$. Hence $\langle f_1, \ldots, f_k \rangle(e_1) \sqsubseteq \langle f_1, \ldots, f_k \rangle(e_2)$.

   Continuity :

   $$\begin{aligned}
   \langle f_1, \ldots, f_k \rangle(\bigsqcup_n e_n) &= (f_1(\bigsqcup_n e_n), \ldots, f_k(\bigsqcup_n e_n)) && \text{[by definition of tupling]} \\
   &= (\bigsqcup_n f_1(e_n), \ldots, \bigsqcup_n f_k(e_n)) && \text{[component functions are continuous]} \\
   &= \bigsqcup_n (f_1(e_n), \ldots, f_k(e_n)) && \text{[LUB of tuples]} \\
   &= \bigsqcup_n \langle f_1, \ldots, f_k \rangle(e_n) && \text{[by definition of } \langle f_1, \ldots, f_k \rangle(e_n)\text{]}
   \end{aligned}$$

2. *Projection:* Define the function $\pi_i : (D_1 \times \cdots \times D_k) \to D_i$, where $\pi_i(d_1, \ldots, d_k) = d_i$. π_i is continuous if all its components, D_j, are cpo's. [Easy proof, left as an exercise.]

3. *Product:* If $f_1 : E_1 \to D_1, \ldots, f_k : E_k \to D_k$ are continuous, then the product $f_1 \times \ldots \times f_k$ is also continuous.

   $$f_1 \times \ldots \times f_k : (E_1 \times \ldots \times E_k) \to (D_1 \times \ldots \times D_k)$$
   $$(f_1 \times \ldots \times f_k)(e_1, \ldots, e_k) = (f_1(e_1), \ldots, f_k(e_k))$$

   In order to prove this, we just need to realize that $(f_1 \times \ldots \times f_k) = \langle f_1 \circ \pi_1, \ldots f_k \circ \pi_k \rangle$. This equality is enough to show that the product function is continuous because we know that the π_i's are continuous, $\circ$ (composition) is continuous, and tupling is continuous.

4. Given the function $h : E \to (D_1 \times \cdots \times D_k)$, then h is continuous $\iff \forall i,\ \pi_i \circ h$ is continuous.

   Proof: "$\Rightarrow$" Because π_i's are continuous and composition preserves continuity.

   "$\Leftarrow$" Because $h = \langle \pi_i \circ h, \ldots, \pi_k \circ h \rangle$ and tupling preserves continuity.

5. Given the function $h : (E_1 \times \ldots \times E_k) \to D$, then

$$\begin{array}{lll} h \text{ is continuous} & \iff & \forall i \,.\, \forall e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_k \,.\, f_i \text{ is continuous, where:} \\ & & f_i : E_i \to D \\ & & f_i(e_i) = h(e_1, \ldots, e_k) \end{array}$$

   That is, h is continuous iff it is continuous on each of its arguments. In other words, to prove that h is continuous, fix all but one element, then prove it is continuous on that element.

   Proof:

   "$\Rightarrow$" We can write f_i as the composition of functions:

$$f_i(e_i) = h(e_1, \ldots, e_k) = (h \circ g)(e_i)$$

   where $g = (g_1, \ldots, g_k)$ and the functions g_i are defined as:

$$\begin{array}{rcl} g_j & : & E_i \to E_j \\ g_j(e_i) & = & e_j \end{array}$$

   Hence $g_j, j \neq i$ are constant functions and g_i is the identity function. We have k continuous functions, so we tuple them and get g continuous. Therefore, $h \circ g = f_i$ is continuous.

   "$\Leftarrow$" For simplicity's sake, assume that h only has two arguments.

$$h : (E_1 \times E_2) \to D$$

   Assume that h is continuous on its first and second arguments.

$$h(\bigsqcup_n (x_n, y_n)) = h(\bigsqcup_p x_p, \bigsqcup_q y_q)$$

   If we regard h as a function of the first argument (so $\bigsqcup_q y_q$ is constant), then we can pull out $\bigsqcup_p$.

$$= \bigsqcup_p h(x_p, \bigsqcup_q y_q)$$

   If we regard h as a function of the second argument (x_p is constant), then we can pull out $\bigsqcup_q$.

$$= \bigsqcup_p \bigsqcup_q h(x_p, y_q)$$

   We claim the following (which proves that h is continuous):

$$\bigsqcup_p \bigsqcup_q h(x_p, y_q) = \bigsqcup_n h(x_n, y_n)$$

   The proof of the claim is by the following lemma.

Lemma. If $\{e_{mn} \mid m \in \mathbb{N}, n \in \mathbb{N}\}$ s.t. $e_{mn} \sqsubseteq e_{m'n'}$ whenever $m \leq m'$ and $n \leq n'$ then

$$\bigsqcup_m \bigsqcup_n e_{mn} = \bigsqcup_n \bigsqcup_m e_{mn} = \bigsqcup_n e_{nn}$$

Proof. We can draw the following diagram to graphically show all of the following least upper bounds: $\bigsqcup_n e_{mn}$, $\bigsqcup_m \bigsqcup_n e_{mn}$, $\bigsqcup_m e_{mn}$, $\bigsqcup_n \bigsqcup_m e_{mn}$, and $\bigsqcup_n e_{nn}$.

$$\begin{array}{cccccccccc}
m=0 & e_{00} & \sqsubseteq & e_{01} & \sqsubseteq & e_{02} & \sqsubseteq & \cdots & \bigsqcup_n e_{0n} \\
 & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & \rotatebox[origin=c]{-45}{$\sqsubseteq$} & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} \\
m=1 & e_{10} & \sqsubseteq & e_{11} & \sqsubseteq & e_{12} & \sqsubseteq & \cdots & \bigsqcup_n e_{1n} \\
 & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & \rotatebox[origin=c]{-45}{$\sqsubseteq$} & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} \\
m=2 & e_{20} & \sqsubseteq & e_{21} & \sqsubseteq & e_{22} & \sqsubseteq & \cdots & \bigsqcup_n e_{2n} \\
 & & & & & & & & \downarrow \\
 & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & & \rotatebox[origin=c]{-90}{$\sqsubseteq$} & \rotatebox[origin=c]{-45}{$\sqsubseteq$} & & \bigsqcup_n \bigsqcup_n e_{mn} \\
 & & & & & & & \ddots & \\
 & \vdots & & \vdots & & \vdots & \vdots & & \bigsqcup_n e_{nn} \\
 & \bigsqcup_m e_{m0} & \sqsubseteq & \bigsqcup_m e_{m1} & \sqsubseteq & \bigsqcup_m e_{m2} & \sqsubseteq & \to & \bigsqcup_n \bigsqcup_m e_{mn}
\end{array}$$

We will show $\bigsqcup_m \bigsqcup_n e_{mn} = \bigsqcup_n e_{nn}$, leaving the rest as an exercise.

Let $x = \bigsqcup_n e_{nn}$. The definition of least upper bound implies $\forall m \,.\, e_{mm} \sqsubseteq x$. Now consider an element e_{mn}, for some m, n. Let $n' = \max(m, n)$. Then $e_{mn} \sqsubseteq e_{n'n'} \sqsubseteq x$. So, $\forall m \,.\, \bigsqcup_n e_{mn} \sqsubseteq x$. Hence, $\bigsqcup_m \bigsqcup_n e_{mn} \sqsubseteq x$. Therefore, $\bigsqcup_m \bigsqcup_n e_{mn} \sqsubseteq \bigsqcup_n e_{nn}$.

Next, let $y = \bigsqcup_m \bigsqcup_n e_{mn}$. This implies $\forall m \,.\, \bigsqcup_n e_{mn} \sqsubseteq y$, which, in turn, implies $\forall m \,.\, \forall n \,.\; e_{mn} \sqsubseteq y$. In particular $\forall n' \,.\, e_{n'n'} \sqsubseteq y$. Therefore, $\bigsqcup_{n'} e_{n'n'} \sqsubseteq y = \bigsqcup_m \bigsqcup_n e_{mn}$.

We have proved that $\bigsqcup_n e_{nn} \sqsubseteq \bigsqcup_m \bigsqcup_n e_{mn}$ and $\bigsqcup_m \bigsqcup_n e_{mn} \sqsubseteq \bigsqcup_n e_{nn}$. By applying antisymmetry of partial orders, we get $\bigsqcup_n e_{nn} = \bigsqcup_m \bigsqcup_n e_{mn}$. The proof for $\bigsqcup_n e_{nn} = \bigsqcup_n \bigsqcup_m e_{mn}$ follows the same path. $\square$