Scribe: Ashwin Machanavajjhala and Vivek Vishnumurthy Lecturer: Radu Rugina

1 Complete Partial Orders

Definition 1.1 (Review: Partial Order Set or Poset)
A partial order set or a poset is a pair $(S, \sqsubseteq)$ consisting of a set S and a binary order relation $\sqsubseteq$ on that set, such that the order relation $\sqsubseteq$ is:

1. Reflexive: $\forall x \in S \,.\, x \sqsubseteq x$.
2. Antisymmetric: $\forall x, y \in S \,.\, (x \sqsubseteq y \land y \sqsubseteq x) \;\Rightarrow\; x = y$.
3. Transitive: $\forall x, y, z \in S \,.\, (x \sqsubseteq y \land y \sqsubseteq z) \;\Rightarrow\; x \sqsubseteq z$

Definition 1.2 (Upper Bounds)
Consider a partial order $(S, \sqsubseteq)$ and a set $X \subseteq S$. An element $p \in S$ is an upper bound of X, if $\forall q \in X,\ q \sqsubseteq p$. An element $p \in S$ is a least upper bound of X if,

1. *p is an upper bound of X, and*
2. *$\forall q,\ q$ an upper bound, $p \sqsubseteq q$.*

The least upper bound *(LUB) of X is denoted as $\bigsqcup X$ and is read as* join of X.

Definition 1.3 (Chains and ω-chains)
Any totally ordered subset of S is called a chain. A chain of the form $x_0 \sqsubseteq x_1 \sqsubseteq \ldots \sqsubseteq x_i \sqsubseteq \ldots$ is called an ω-chain.

Definition 1.4 (COMPLETE PARTIAL ORDERS)
A partial order set $(S, \sqsubseteq)$ is called a complete partial order (cpo) if every ω-chain has a LUB.

Let us look at some of the examples of posets we had considered in the previous lecture and investigate whether they are cpos.

- $(\mathcal{P}(X), \subseteq)$: *Yes.*
  This *is a cpo* since the LUB of any set in this poset is the set itself, i.e., $\bigsqcup \mathcal{P}(X) = X$. Note that in this case, the join operator $\bigsqcup$ is nothing but the union operator $\bigcup$.
- $(\mathbb{N}, \leq)$: *No.*
  Since there is no LUB for the following ω-chain: $0 \leq 1 \leq 2 \leq \ldots$
- $(\mathbb{N} \cup \{\infty\}, \leq)$, where $\forall n \in \mathbb{N},\ n \leq \infty$: *Yes.*
  Now every ω-chain has a LUB. In particular, ∞ is the LUB of the chain $0 \leq 1 \leq 2 \leq \ldots$ (which can be written as $\bigsqcup \mathbb{N} \cup \{\infty\} = \infty$). For this cpo the join operator is the maximum function.
- Similarly, $(\mathbb{R}, \leq)$ is *not* a cpo, while $(\mathbb{R} \cup \{\infty\}, \leq)$ *is* a cpo.
- $(\mathbb{Q} \cup \{\infty\}, \leq)$: *No.* This is because an increasing sequence of rational numbers might converge into a *hole*, i.e., an irrational number. For example, consider the increasing sequence, $x_0 = 1, x_{n+1} = \frac{3x_n+4}{2x_n+3}$. One can work out that this sequence converges to $\sqrt{2} \notin \mathbb{Q}$.
- $(S, =)$: *Yes.*
  All of the ω-chains of this cpo are of the form $x = x = \ldots$ with x being the LUB.

- $(S, \sqsubseteq)$ is a poset $\Leftrightarrow$ $(S, \sqsupseteq)$ is a poset. However, $(S, \sqsubseteq)$ is a cpo $\not\Leftrightarrow$ $(S, \sqsupseteq)$ is a cpo.
  For example, though $(\mathbb{R} \cup \{\infty\}, \leq)$ is a cpo since every ω-chain has a LUB, $(\mathbb{R} \cup \{\infty\}, \geq)$ is *not* a cpo since there is no LUB for a sequence of non-increasing real numbers. This arises due to the asymmetric definition of cpos – it talks only about ascending chains.

- $(B, \sqsubseteq_B)$ is a poset $\Rightarrow$ $(A \to B, \sqsubseteq)$ is a poset, where $f_1 \sqsubseteq f_2$ iff $\forall x \in A, f_1(x) \sqsubseteq_B f_2(x)$.
  Does $(B, \sqsubseteq_B)$ being a cpo imply that $(A \to B, \sqsubseteq)$ is a cpo? Yes.

  We have to show that for every ω-chain in $A \to B$, $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$, there exists a function $g = \bigsqcup_{n \in \mathbb{N}} f_n$. Let x be some arbitrary value of A. Then, by the definition of $\sqsubseteq$ for the domain $A \to B$, the following is an ω-chain in B: $\omega_1 \;=\; f_1(x) \sqsubseteq_B f_2(x) \sqsubseteq_B \ldots$. Since $(B, \sqsubseteq_B)$ is a cpo, this ω-chain has a LUB. Let $\bigsqcup_{n \in \mathbb{N}} f_n(x)$ be the LUB of ω_1 and define $g(x) = \bigsqcup_{n \in \mathbb{N}} f_n(x)$.

  It can be proven that the function g, defined as above for every x, is a LUB for $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ (proof left as an exercise).

1.1 Complete Partial Orders with Bottom

Definition 1.5 (CPO with Bottom) *A triple* $(S, \sqsubseteq, \perp)$ *is called a* complete partial order with bottom *if it contains an element* $\perp \in S$ *such that* $\forall x \in S \,.\, \perp \sqsubseteq x$.

If $(B, \sqsubseteq_B)$ is a cpo with bottom $\perp_B$, then $(A \to B, \sqsubseteq)$ is a cpo with bottom $\perp_{A \to B} = \lambda x.\perp_B$

2 Lattices

We next clarify the distinction between cpo's/cpo's with bottom (used in denotational semantics) to lattices/complete lattices (used in dataflow analysis), as both of them are defined using partial orders.

Definition 2.1 (Lattices)
A poset $(S, \sqsubseteq)$ *is a Lattice if: any finite* $X \subseteq S$ *has a Least Upper Bound (LUB) and a Greatest Lower Bound (GLB)*[1].

Definition 2.2 (Complete Lattices)
A poset $(S, \sqsubseteq)$ *is a Complete Lattice if: any* $X \subseteq S$ *has a Least Upper Bound (LUB) and a Greatest Lower Bound (GLB).*

(Note: Complete Lattices do not have the restriction of finite sets)

The distinction between cpo's and lattices is that in cpo's we talk about ω-chains, whereas in lattices we speak of arbitrary finite/infinite sets. The diagram below shows the relation between cpo's and lattices, where the arrows indicate implication.

$$
\begin{tikzcd}
 & (S, \sqsubseteq) \text{ cpo with } \perp \arrow[rr] & & (S, \sqsubseteq) \text{ cpo} \arrow[dr] & \\
(S, \sqsubseteq) \text{ complete lattice} \arrow[ur] \arrow[drr] & & & & (S, \sqsubseteq) \text{ poset} \\
 & & (S, \sqsubseteq) \text{ lattice} \arrow[urr] & &
\end{tikzcd}
$$

[1]The Greatest Lower Bound is defined similarly to the Lowest Upper Bound

Examples

- ({0,1}, =) is a cpo, but not a lattice, because {0,1} has no LUB.
- $(\mathbb{Q}, \leq)$ is a lattice, but not a cpo. (Any finite subset of rational numbers has a maximum, which is the LUB of the set)
- If S is finite, $(S,\sqsubseteq)$ is a lattice $\Leftrightarrow$ $(S,\sqsubseteq)$ is a complete lattice.
- If S is finite, $(S,\sqsubseteq)$ is a partial order $\Leftrightarrow$ $(S,\sqsubseteq)$ is a complete partial order.

3 Background to the Fixed Point Theorem

In Denotational Semantics, we talked of functions in $\Sigma \to \Sigma_\perp$. Let's check if this is a cpo domain.

- $(\Sigma, =)$ is a cpo.
- $(\Sigma_\perp, \sqsubseteq_1)$ is a cpo with $\perp$. Here $\sigma \sqsubseteq_1 \sigma'$ iff $(\sigma = \sigma') \vee (\sigma = \perp)$

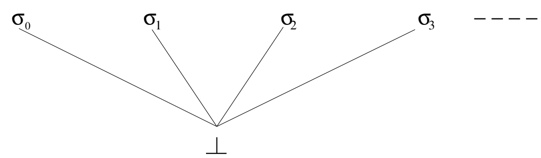


Figure 1: The Hasse Diagram

- $(\Sigma \to \Sigma_\perp, \sqsubseteq_2)$ is a cpo with bottom. Here $f \sqsubseteq_2 f'$ iff $\forall \sigma \in \Sigma \cdot f(\sigma) \sqsubseteq_1 f'(\sigma)$
  and $\perp_{\Sigma \to \Sigma_\perp} = \lambda\sigma \in \Sigma \cdot \perp$ (This function returns $\perp$ for any element in Σ)

Let's go back to the while loop $w =$ **while** b **do** c and its Denotational Semantics.
$\mathcal{C}[\![w]\!] = \bigsqcup_{n \in N} \Gamma^n(\perp_{\Sigma \to \Sigma_\perp}) = fix(\Gamma)$

$$\begin{aligned} \text{where } \Gamma(\gamma) = & \{(\sigma,\sigma) \mid \neg\mathcal{B}[\![b]\!]\sigma\} \cup \\ & \{(\sigma,\perp) \mid \mathcal{B}[\![b]\!]\sigma \wedge \mathcal{C}[\![c]\!]\sigma = \perp\} \cup \\ & \{(\sigma,\gamma(\mathcal{C}[\![c]\!])) \mid \mathcal{B}[\![b]\!]\sigma \wedge \mathcal{C}[\![c]\!]\sigma \neq \perp\} \end{aligned}$$

Here : $\Gamma : (\Sigma \to \Sigma_\perp) \to (\Sigma \to \Sigma_\perp)$

From the lecture on Inductive Definitions, we know that $\bigsqcup_{n \in N} \Gamma^n(\perp_{\Sigma \to \Sigma_\perp})$ is the least fixed point of the recursive definition for Γ. We would like to know what is the fundamental property of Γ that allow us to talk about its least fixed point? In the Inductive Definitions lecture we talked about monotonicity. Is that necessary to guarantee the existence of the least fixed point? Is that sufficient?

3.1 Monotonicity

Consider the function $f : \{0,1\} \to \{0,1\}$ and the poset $(\{0,1\}, =)$, where

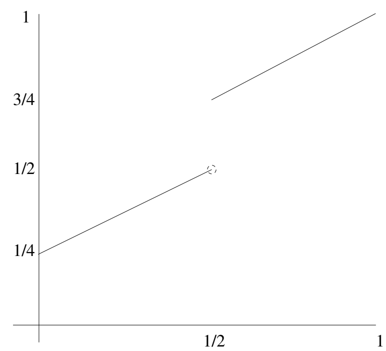
- $f(0) = 1$
- $f(1) = 0$

f is not monotonic. Now what is $\bigsqcup_{n \in N} f^n(0)$?
The sequence $f^n(0)$ is: *0, 1, 0, 1, 0 ...* This sequence has no LUB. Thus we cannot build an increasing sequence (an ω-chain) of the form $\{f^n(0)\}_n$ without monotonicity and we cannot talk about the least upper bound $\bigsqcup f^n(0)$. So, a function f has to be monotonic if it is to have a least fixed point of the form $\bigsqcup f^n(0)$.

3.2 Continuity

Now consider a function f and $([0,1], \leq)$: $f(x) = \frac{2x+1}{4}, x < \frac{1}{2}$

$$f(x) = \frac{2x+2}{4}, x \geq \frac{1}{2}$$



It is easy to check that this function is monotonic. We want to know if $\bigsqcup_{n\in\mathbb{N}} f^n(0)$ is its least fixed point.

The sequence $f^n(0)$ is: $0 \sqsubseteq \frac{1}{4} \sqsubseteq \frac{1}{4} + \frac{1}{8} \sqsubseteq \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \sqsubseteq \dots$
(Note: this sequence is an ω-chain because f is monotonic)

This sequence converges to $\frac{1}{2}$. i.e., $\bigsqcup_{n\in\mathbb{N}} f^n(0) = \frac{1}{2}$.

Is this is a fixed point? No, because $f(\frac{1}{2}) = \frac{3}{4} \neq \frac{1}{2}$

The least fixed point of this function is 1.

Therefore $\bigsqcup_{n\in\mathbb{N}} f^n(0)$ is not the least fixed point for this function (it is not even a fixed point). The reason why this is not the least fixed point is that the function is not *continuous*.

Definition 3.1 (Continuous functions)
If $(D, \sqsubseteq)$ is a cpo and $f : D \to D$ is monotonic, then f is continuous if for any ω-chain $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \dots$ we have:

$$f(\bigsqcup_{n\in\mathbb{N}} x_n) = \bigsqcup_{n\in\mathbb{N}} f(x_n)$$

Note: on a finite domain, a function is monotonic iff it is continuous.

3.3 The Fixed Point Theorem

If $(D, \sqsubseteq, \perp)$ is a cpo with bottom, $f : D \to D$ is a continuous function, and we define:

$$\mathit{fix}(f) = \bigsqcup_{n\in N} f^n(\perp)$$

then:

- $\mathit{fix}(f)$ is a fixed point of f.
- $\mathit{fix}(f)$ is the least fixed point of f.