What to turn in

Turn in the written assignment during class on the due date.

Problem 1: Partial Orders

1a) Consider a relation $\sqsubseteq$ over natural numbers $\mathbb{N}$ defined as follows:

$$\begin{array}{lcl} \forall k \geq 1 & . & 3k-1 \sqsubseteq 3k \\ \forall k \geq 0 & . & 3k \sqsubseteq 3k+1 \\ \forall k \geq 0 & . & 3k+2 \sqsubseteq 3k \\ \forall k \geq 1 & . & 3k \sqsubseteq 3k-2 \end{array}$$

Prove that this relation is antisymmetric.

1b) Given an arbitrary set S and a relation R, consider the reflexive, transitive closure of R as a new relation R^* over S, inductively defined as follows:

$$\begin{array}{l} \text{if } x = y \text{ then } x \ R^* \ y \\ \text{if } x \ R \ y \text{ then } x \ R^* \ y \\ \text{if } (\exists z \in S \ . \ (z \neq x) \wedge (z \neq y) \wedge (x \ R^* \ z) \wedge (z \ R^* \ y)) \text{ then } x \ R^* \ y \end{array}$$

In general, the reflexive, transitive closure of an antisymmetric relation is not necessarily antisymmetric. Give an example of a set and an antisymmetric relation R such that R^* is not antisymmetric.

1c) Show that the reflexive and transitive closure $\sqsubseteq^*$of the relation defined in part 1a) is antisymmetric.

1d) Hence, ($\mathbb{N}$, $\sqsubseteq^*$) is a partial order. Argue whether it is a complete partial order and whether it has a bottom element. You must prove your results.

1e) Given a partial order $(S, \sqsubseteq)$, we say that $S' \subseteq S$ is an *anti-chain* if there is no pair of distinct elements in S' which are comparable with respect to $\sqsubseteq$. Further, we can define the *width* of $(S, \sqsubseteq)$ as the cardinality (number of elements) of its largest anti-chain. Note that there may be multiple largest anti-chains for a given partial order.

Write two largest anti-chains for the partial order ($\mathbb{N}$, $\sqsubseteq^*$) defined above and compute the width of this poset.

Problem 2: Continuous functions

2a) Consider $(D, \sqsubseteq_1)$ a cpo, $(D_\perp, \sqsubseteq_2, \perp_D)$ the corresponding lifted cpo, and $(E, \sqsubseteq_3, \perp_E)$ another cpo with bottom. Show that if $f : D \to E$ is a continuous function, then the function f^* defined in class:

$$\begin{array}{rcl} f^* & : & D_\perp \to E \\ f^*(d) & = & \begin{cases} f(d') & \text{if } \ d = \lfloor d' \rfloor \\ \perp_E & \text{otherwise} \end{cases} \end{array}$$

is continuous.

2b) Show that the higher-order function:

$$\begin{aligned} g &: [D \to E] \to [D_\perp \to E] \\ g(f) &= f^* \end{aligned}$$

is also continuous.

2c) By the Fixed Point Theorem, each continuous function has a least fixed point. Hence, for any complete partial order domanin $(D, \sqsubseteq)$ with bottom $\perp$, the least fixed point operator:

$$\begin{aligned} \mathit{fix} &: [D \to D] \to D \\ \mathit{fix}(f) &= \text{least fixed point of } f \end{aligned}$$

is well-defined. Show that *fix* is continuous.

Problem 3: Denotational Semantics

Consider the following two **IMP** programs, written using the concrete syntax from the last assignment to avoid syntactic ambiguities:

```
P1:   Y := 0;
      while (1 <= N) {
         Y := Y + M;
         N := N - 1
      }

P2:  if (1 <= N) then {
         Y := M * N;
         N := 0
      } else {
         Y := 0
      }
```

3a) The denotational semantics of the while loop in the first program is the least fixed point of a higher order function Γ. Write the first three approximations of the loop semantics, $\Gamma(\perp)$, $\Gamma^2(\perp)$, and $\Gamma^3(\perp)$.

3b) Use denotational semantics to show that these two programs are equivalent.