Ordinal Representations

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What Is An Ordinal Representation?

Definition. A representation of a preference relation $\succ$ on X is a function $U : X \to \mathbb{R}$ such that $x \succ y$ iff $U(x) > U(y)$.

Representations of Relations

Example 1. Temperature assigns numbers to locations that represents the "hotter than" order.
$X = \{\text{Boston, New York, Miami,} \ldots\}$. $T(\text{Miami}) > T(\text{Boston})$ iff Miami is hotter than Boston.

$$T(\text{Miami}) - T(\text{Boston}) > T(\text{New York}) - T(\text{Toronto})$$

says that the gain in heat going from Boston to Miami is greater than the gain in heat in going from Toronto to New York.
Temperature is an example of an interval scale of measurement.

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Representations of Relations

Example 2. Height assigns numbers to people that represents the "taller than" order. $X = \{\text{Ali, Bill, Krishna, Sue,} \ldots\}$.

$$H(\text{Bill}) - H(\text{Sue}) > H(\text{Ali}) - H(\text{Krishna}) \text{ and } H(\text{Bill}) = 2H(\text{Mini-Me})$$

are all meaningful statements. Height is an example of a ratio scale.

Representing Scales?

Scales can be described by their invariance properties: What are the class of functions that represent a particular scale?

Temperature scales

$$\deg F = \frac{9}{5} \deg C + 32$$

Temperature is an *interval scale* because if $T : X \rightarrow \mathbb{R}$ represents temperature, then $T' : X \rightarrow \mathbb{R}$ also represents temperature iff there is an $a > 0$ and a b such that $T'(X) = aT(x) + b$.

Representing Scales?

Scales can be described by their invariance properties: What are the class of functions that represent a particular scale?

Height scales

$$\text{inches } I = 12 \cdot \text{ feet } F$$

Height is a *ratio scale* because if $T : X \to \mathbb{R}$ represents height, then $T' : X \to \mathbb{R}$ also represents height iff there is an $a > 0$ such that $T'(X) = aT(x)$.

What Is An Ordinal Representation?

Utility is an ordinal scale.

Proposition. $U : X \to \mathbb{R}$ and $V : X \to \mathbb{R}$ both represent the same preference relation $\succ$ iff there is a strictly increasing function $f : \text{range}\, U \to \mathbb{R}$ such that for all $x \in X$, $V(x) = f \circ U(x) \equiv f\big(U(x)\big)$.

Proof.

$\Rightarrow$ If f is strictly increasing, then $x \succ y$ iff $U(x) > U(y)$ iff $f(U(x)) > f(U(y))$.

$\Leftarrow$ The function $f(r) = V(U^{-1}(r))$ is well-defined on the range of U because although U need not be 1 to 1, if $U(x) = U(y)$ then $V(x) = V(y)$. If $s > r$, $x \in U^{-1}(s)$ and $y \in U^{-1}(r)$, then $x \succ y$, and so

$$f(s) = V(U^{-1}(s)) = V(x) > V(y) = V(U^{-1}(r)) = f(r).$$

Finally, $f(U(x)) = V(U^{-1}(U(x)) = V(x)$. ■

Existence Of Ordinal Representations

- Does every preference relation have a representation? More generally, what binary relations have numerical representations?

- Does every function from X to $\mathbb{R}$ represent some preference relation? That is, for a given $U : X \to \mathbb{R}$, define $x \succ_U y$ iff $U(x) > U(y)$. Is $\succ_U$ a preference relation?

The Denumerable Case

Proposition. Suppose X is denumerable.

a. If $\succ$ is a preference relation on X, then it has a utility representation.

b. If $U : X \to \mathbb{R}$ then $\succ_U$ is a preference relation.

Proof. b. U(x)>U(y) implies $U(y) \not> U(x)$, so $\succ_U$ is asymmetric. If $x \not\succ_U y$ and $y \not\succ_U z$, then $U(y) \geq U(x)$ and $U(z) \geq U(y)$. Thus $U(z) \geq U(x)$, and so $x \not\succ_U z$.

a. Index X: $X = \{x_1, x_2, \ldots\}$. For each $x \in X$, define the "no better than" set $W(x) = \{w : w \not\succ x\}$, and the set of "no better than" indices $N(x) = \{n : x_n \in W(x)\}$. Finally, define

$$U(x) = \sum_{n \in N(x)} \left(\frac{1}{2}\right)^n .$$

The Denumerable Case

Suppose that $x \succ y$. If $w \in W(y)$ then $w \nsucc y$ and $y \nsucc x$, so NT implies $w \nsucc x$. Thus $W(y) \subset W(x)$. A implies that $x \in W(x)$ and that $x \notin W(y)$, so $W(y) \subsetneq W(x)$. Consequently $N(y) \subsetneq N(x)$, and so $U(x) > U(y)$.

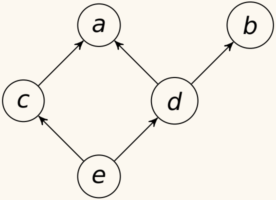
Suppose that $U(x) > U(y)$. There are only three possibilities: $x \succ y$, $x \sim y$, and $y \succ x$. We will rule out the last two.

The third is ruled out because we have already shown that $y \succ x$ implies $U(y) > U(x)$. The second is ruled out because if $x \sim y$, then $x \in W(y)$ and $y \in W(x)$. NT would imply that $W(x) = W(y)$ and thus $U(x) = U(y)$, contradicting the hypothesis. This leaves $x \succ y$. ■

Krep's proof has a small bug. See if you can find it.

Partial Orders

Here is a partial order.

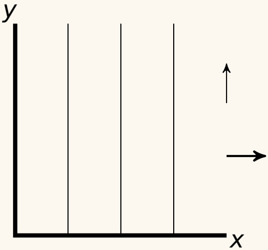


If $\succ$ had an ordinal representation U, we would have $U(a) = U(b)$, $U(b) = U(c)$ and $Ua) > U(c)$, which is impossible.

Definition. A weak representation of the partial order $\succ$ is a function $U : X \to \mathbb{R}$ such that if $x \succ y$, then $U(x) > U(y)$.

The Uncountable Case

Example. Let $X = \mathbb{R}^2_+$. Define $(x_1, y_1) \succ (x_2, y_2)$ if either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$. This is the lexicographic order on $\mathbb{R}^2_+$.



If $\succ$ had a utility representation U, then for all x $U(x, b_2) > u(x, a_2)$, and if $x' > x$, $U(x', a_2) > U(x, b_2)$. Between each $U(x, b_2)$ and $u(x, a_2)$ there is a rational number. Since there are an uncountable number of x values, there would need to be an uncountable number of rationals, a contradiction. Thus $\succ$ has no utility representation.

The Uncountable Case

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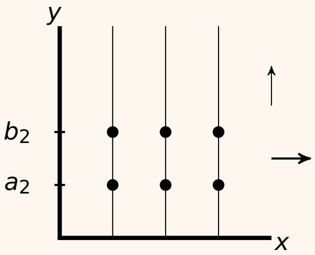


If $\succ$ had a utility representation U, then for all x $U(x, b_2) > u(x, a_2)$, and if $x' > x$, $U(x', a_2) > U(x, b_2)$. Between each $U(x, b_2)$ and $u(x, a_2)$ there is a rational number. Since there are an uncountable number of x values, there would need to be an uncountable number of rationals, a contradiction. Thus $\succ$ has no utility representation.

The Uncountable Case

How do the rationals sit in $\mathbb{R}$? There are only a countable number of rational numbers, and between every two distinct reals there is a rational number. The rational numbers are order-dense in the reals.

Definition. A set $Z \subset X$ is order-dense if for each pair of elements $x, y \in X$ such that $x \succ y$ there is a $z \in Z$ such that $z \not\succ x$ and $y \not\succ z$.

Proposition. A preference order $\succ$ on X has a utility representation iff X contains a countable order-dense subset.

This proposition is due to Georg Cantor, who gave us the mathematics of infinity.

Proof.
$\Rightarrow$ The denumerable construction basically works. Suppose Z is order-dense. A gap in $\succ$ is a pair of choices x and y such that $x \succ y$ and there is no $w \in X$ such that $x \succ w \succ y$.

The Uncountable Case

First observe that if $\succ$ admits an order-dense subset and if $x \succ y$ is a gap, then one of x and y must be in Z. Thus since Z is countable, the number of gaps is countable. Define the set Z' to be the union of Z and all gap pairs, together with a maximal element on X if one exists and a minimal element of X if one exists. This set is countable.

Now proceed as before. Index Z' and define $N(x)$ to be the indices of elements in $W(x) \cap Z$. Use the utility function of slide 8 and check that it works.

If U represents $\succ$ and $U(X)$ is an interval I, the proof of the reverse implication is easy — just take $Z = U^{-1}(I \cap \mathbb{Q})$. Things get tricky if $U(X)$ is, say a Cantor set or some other weird set. The proof here accounts for that, but it really adds nothing to our intuition.

The Uncountable Case

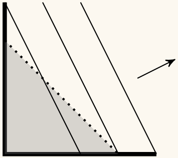
$\Leftarrow$ We will construct Z. Let R denote the set of all pair of rational numbers r, r' such that $r > r'$. R is a subset of $\mathbb{Q} \times \mathbb{Q}$, and so is countable. For each (r, r') pair in R, if $U^{-1}\big((r, r')\big)$ is non-empty, choose one z such that $r > U(z) > r'$. The set Z' of all such z is countable. Add to this set all gap pairs to get set Z. The number of gap pairs is countable, and so Z is countable.

Z is also order-dense. To see this, choose $x \succ y$. Then $U(x) > U(y)$. If x, y is a gap pair, then both $z = x$ and $z = y$ satisfy the criterion of order-denseness. If they are not a gap pair, there is a w such that $x \succ w \succ y$. Then there exists rational numbers $r > r'$ such that $U(x) > r > U(w) > r' > U(y)$. By construction there is a $z \in Z$ such that $r > U(z) > r'$, so $x \succ z \succ y$. Asymmetry implies that $z \not\succ x$ and $y \not\succ z$, and so the order-denseness criterion is again satisfied. ∎

Choice

Example 1. $X = \mathbb{R}^2_+$ $\qquad$ $A = \{(x, y) : x + y < 1\}$

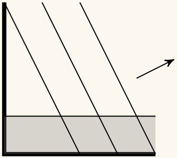
$$U(x, y) = 2x + y$$



Choice

Example 2. $X = \mathbb{R}^2_+$ $\qquad A = \{(x, y) : y \leq 1\}$

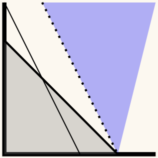
$$U(x, y) = 2x + y$$



Choice

Example 3. $X = \mathbb{R}^2_+$ $\qquad A = \{(x,y) : x + y \leq 1\}$

$$U(x,y) = \begin{cases} x + 2y & \text{if } 2x + y < 1, \\ 0 & \text{otherwise.} \end{cases}$$



Choice — Final Facts

When will choice exist for sets of infinite cardinality?

- A set $A \subset \mathbb{R}^n$ is compact if it is both closed and bounded.
- If U is continuous, then it has a maximum value on every compact set.

When does a preference relation have a continuous representation?

Definition. A preference relation is continuous if $\{(x, y) \in X \times X : x \succ y\}$ is open.

This expresses the idea that if $x \succ y$ if x' is near to x and y' is near to y, then $x' \succ y'$.

Proposition. A preference order has a continuous utility representation iff it is continuous.

The proof is difficult because of the possibility of gaps. See Debreu, G. "Continuity properties of Paretian utility." *International Economic Review* 5, 285-293 (1964).