CS 5846 & Econ 3810/6760
HW Set #3 — Sp 2021

1. Consider the set of lotteries (p_x, p_y, p_z) on the set of outcomes $\{x, y, z\}$ where p_x, p_y, and p_z are the probabilities of x, y, and z, respectively. For each (partial) preference below, determine whether it is consistent with expected utility maximization. (If yes, find a utility function; if no, show that it cannot come from an expected utility maximizer.)
   (a) $(0, 1, 0) \succ (1/8, 6/8, 1/8)$ and $(7/8, 0, 1/8) \succ (6/8, 1/8, 1/8)$.
   (b) $(1/4, 1/4, 1/2) \succ (3/4, 0, 1/4) \succ (5/6, 1/6, 0) \succ (1/2, 1/3, 1/6)$.

2. You have an opportunity to place a bet on the outcome of an upcoming race involving a certain horse, named Bayes, at even odds: That is, if you bet x dollars and Bayes wins, you will have $w_0 + x$, while if she loses you will have $w_0 - x$, where w_0 is your initial wealth.
   (a) Suppose that you believe Bayes will win with probability p and that your utility for wealth w is $\ln(w)$. Find your optimal bet as a function of p and w_0. What does it mean if x is negative?
   (b) You know little about horse racing, only that racehorses are either winners or average, that winners win 90% of their races, and that average horses win only 10% of their races. After all the buzz you've been hearing, you are 90% sure that Bayes is a winner. What fraction of your wealth do you plan to bet?
   (c) You run into a friend who spends a lot of time at the track. He can accurately guess a horse's true type (i.e., whether he's a winner or just average) 95% of the time (he is wrong only 5% of the time). He says,"despite what you may think, Bayes is only average." How much will you now bet?

3. Juana has w dollars and CARA utility with parameter $\alpha > 0$. With some probability $0 < q < 1$ she will get sick, and will need to spend D dollars on her recovery. She has the opportunity to buy and insurance policy that will cost some fixed amount p and will cover all of her expenses.
   (a) Find the set of prices she is willing to pay in order to have the insurance. What is the maximum price p^* she is willing to pay?
   (b) [GRAD] How does it depend on w, D, q, and α? (That is, how does p^* changes as each of these parameters increases.)
   (c) Suppose there is a test t that reports back either $+$ or $-$. If it reports back a $+$, the probability of her getting sick increases to $q' > q$. If the report is $-$, she will not get sick. For a given policy price $p \leq p^*$, how much is Juana willing to pay to take the test?

4. (Attributed to Richard Zeckhauser) In Russian Roulette, a number of bullets are loaded into a revolver with six chambers; an individual then points the revolver at his head, pulls the trigger, and is killed if and (hopefully) only if the revolver goes off. Assume that

the individual must play this game; that he is an expected-utility maximizer; and that each chamber is equally likely to be in firing position, so that if the number of bullets is b, his probability of being killed is $b/6$. Suppose further that the maximum amount he is willing to pay to have one bullet removed from a gun initially containing only one bullet is $\$x$, and the maximum amount he is willing to pay to have one bullet removed from a gun initially containing 4 bullets is $\$y$, where x and y are both finite. Finally, suppose that he prefers more money to less and that he prefers life (even after paying $\$x$ or $\$y$) to death. Let UD denote his von Neumann-Morgenstern utility when dead, which is assumed to be independent of how much he paid (as suggested by empirical studies of his von Neumann-Morgenstern utilities when alive after paying \$0, $\$x$, or $\$y$ respectively.

(a) What restrictions are placed on U_D, $U_A(0)$, $U_A(x)$, and $U_A(y)$ by the assumption that he prefers more money to less when alive?

(b) What restrictions are placed on U_D, $U_A(0)$, $U_A(x)$, and $U_A(y)$ by the assumption that he prefers life (even after paying $\$x$ or $\$y$) to death?

(c) Is it possible to tell from the information given above whether $x > y$ for an expected utility maximizer? Does it matter whether he is risk-averse? Explain.

5. [GRAD] Suppose that a risk-averse preference relation on simple lotteries in $\mathbb{R}$ has an expected utility representation wherein the payoff function is C^2, that is, twice continuously differentiable. Consider lotteries of the form $l(\delta) = (0.5, \mu - \delta; 0.5, \mu + \delta)$ and show that the certainty equivalent $C(\delta)$ for the lottery $l(\delta)$ has the form

$$CE(\delta) = \mu - \frac{\rho_A(\mu)}{2}\sigma^2 + o(\delta)$$

where $o(\delta)$ is a term such that $\lim_{\delta\to 0} o(\delta)/\delta = 0$, and σ^2 is the variance of the lottery. [Note that the expression has been corrected; it used to say $\rho_A(\delta)$, not $\rho_A(\mu)$.]

6. (a) In the development of the Savage model, show that the conditional order $f \succ_A g$ is a preference relation. (This last part is Kreps' Lemma 9.5. Note that to show that it is a preference order, you have to show that it is well defined, in the following sense: recall that $f \succ_A g$ is defined to hold if $f_A h \succ g_A h$ for some act h. (The definition given in Kreps is equivalent to this.) If there existed acts h and h' such that $f_A h \succ g_A h$ but $g_A h' \succ f_A h'$, then $\succ_A$ would not be well defined. You do have to explain why that doesn't happen.)

   (b) Let the symbols x and y refer both to the outcomes x and y and to the constant acts that pay off x and y, respectively, for sure. (Which we mean will be clear from its context.) For any event E, define $x_E y$ to be the act that pays off outcome x on E and outcome y on E^c. Define the more-likely than relation $A > B$ on events as follows: $A > B$ if there are $x \succ y$ such that $x_A y \succ x_B y$. Show that $>$ is a preference relation; that is, negatively transitive and asymmetric.

7. Suppose that there are two states of the world, $S = \{s, t\}$. Suppose that the bettor will accept both a bet at $2:1$ odds on s and at $2:1$ odds on t. That is, in each case she will

accept a bet where she wins \$1 should the state occur, and loses \$2 otherwise. Give a Dutch book against her, a system of acceptable bets against which she will surely lose. (And you may assume that if she finds a given bet acceptable, she will also accept bets at even more favorable odds.)