Decision Theory I
Problem Set 1

1. Show that if $\succ$ is negatively transitive and asymmetric then $\succ$ is transitive.

2. Suppose $X = \{x, y, z\}$. Consider a choice function such that $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y\}$. Does this choice function satisfy Sen's α and β?

3. The set of alternatives is $X = \{a, b, c\}$ and $\succ$ is a binary order on X reflecting strict preference. Suppose that for $x \in \{b, c\}$, $x \not\succ a$ and $a \not\succ x$. Suppose also that $b \succ c$. Can this relation be a strict preference relation? Explain.

   If we want to include the possibility that there is an alternative a that is not comparable to either b or c in our analysis then we would want the condition above on a to be satisfied. What does this example say about non-comparability? (That is, what properties is non-comparability forced to have if $\succ$ is a preference relation?)

4. Let $\succ$ be a binary relation on a finite set X. Define $\succeq$ by: $x \succeq y$ if $y \not\succ x$. Show

   (a) If $\succeq$ is complete then $\succ$ is asymmetric.

   (b) If $\succeq$ is transitive then $\succ$ is negatively transitive.

5. A binary relation that is reflexive, symmetric and transitive is called an equivalence relation. An equivalence relation partitions a set into equivalence classes. Suppose that $\succ$ is a strict preference relation on X. Then by Proposition 2.4 of Kreps we know that $\sim$ is an equivalence relation on X. For each $x \in X$ define its equivalence class by $I(x) = \{y \in X \mid y \sim x\}$. Show that the sets $I(x)$ partition X and that the sets $I(x)$ are strictly ranked. (A collection of sets $\{A_1, \ldots, A_N\}$ partitions X if each $x \in X$ is in at least one A_i, and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Showing that $I(x)$ is *strictly ranked* means showing that if $I(x) \neq I(y)$, then either $I(x) \succ I(y)$ or $I(y) \succ I(y)$, where $I(x) \succ I(y)$ means that for all $x' \in I(x)$ and all $y' \in I(y)$, we have $x' \succ y'$.)

6. **GRAD:** In the statement of Sen's α and β we allow the sets A and B to be any subsets of X. So when we proved that these axioms imply a rational revealed preference relation we allowed ourselves to use information about choices from arbitrary subsets of X. Find the smallest collection of subsets of X such that if we require α and β to be satisfied on this collection of sets the claim in the revealed preference theorem is true. (I.e., find a relatively small collection $\mathcal{Y}$ of subsets of X such that if we have a choice function C such that Sen's α and β hold for all the sets in $\mathcal{Y}$, but perhaps not for other subsets of X, that's enough to show that C is determined by a preference relation.)

7. **GRAD:** In class, in the proof of the revealed preference theorem we defined strict revealed preference. Weak revealed preference is defined as follows: $x \succeq y$ if $x \in C(\{x, y\})$. Define induced strict revealed preference relation $\succ^*$ using the revealed preference $\succeq$ as follows: $x \succ^* y$ if $x \succeq y$ and $y \not\succeq x$. Suppose the choice function satisfies Sen's α and β. Are strict revealed preference and induced strict revealed preference the same relation? (I.e., are $\succ$ and $\succ^*$ the same? Either prove that they are, or give a counterexample.)