Decision Theory I
Problem Set 3

For all of the problems on this problem set you can assume that the axioms we used in proving the Von Neumann Morgenstern Theorem (Theorem 5.4 in Kreps) are satisfied. Note that by Lemma 5.7 of Kreps these axioms imply that there are best and worst elements of the set of prizes X. You can use this lemma without proof.

1. Suppose that the set of prizes is $X = \{1, 4, 9\}$. Suppose that $p = (1/2, 0, 1/2)$, $q = (1/3, 1/3, 1/3)$ and $r = (1/5, 2/5, 2/5)$ are probabilities on X. The individual's preferences $\succ$ have an expected utility representation with utility function over prizes, $u(x) = x^{1/2}$. How does the individual rank p, q and r?

2. Consider a finite set X of prizes and probabilities $\mathcal{L}$ on them. Suppose that an individual's preferences $\succ$ on $\mathcal{L}$ have an expected utility representation with utility function on prizes $u : X \to \mathbb{R}$. Show that $\succ$ satisfies the independence axiom.

3. Consider a finite set X of prizes and probabilities $\mathcal{L}$ on them. Suppose that an expected utility maximizer's preferences $\succ$ on $\mathcal{L}$ have an expected utility representation with utility function on prizes $u : X \to \mathbb{R}$.

   (a) Suppose that $v(\cdot) = au(\cdot) + b$ for real numbers $a > 0$ and b. Show that v also represents $\succ$.

   (b) **GRAD**: Suppose that $\succ$ has another expected utility representation with another utility function on prizes $v : X \to \mathbb{R}$. Show that there exist real numbers $a > 0$ and b such that $v(\,\cdot\,) = au(\,\cdot\,) + b$.

4. Either prove or provide a counterexample for the following claim. If $\mathbb{E}[\mathbf{x}] = \mathbb{E}[\mathbf{y}]$ and $\mathbb{V}[\mathbf{x}] < \mathbb{V}[\mathbf{y}]$, then every expected-utility maximizer with a strictly increasing and strictly concave utility function would prefer lottery $\mathbf{x}$ to lottery $\mathbf{y}$. Here, $\mathbb{V}[\,\cdot\,]$ denotes the variance operator.

5. An expected utility maximizing investor must decide how much of his initial wealth $w > 0$ to invest in each of two risky asset. The gross return of the assets is $\mathbf{z}_1$ and $\mathbf{z}_2$, respectively. The individual's utility function is u with $u'(w) > 0 > u''(w)$ for all w. Let (α_1^*, α_2^*) denote the optimal investment for the individual.

(a) Would the investor ever find it optimal to invest a positive amount in an asset with negative expected net return? That is, is it possible that $\alpha_1^* > 0$ and $\mathbb{E}[\mathbf{z}_1] < 1$?

(b) Suppose that both assets have positive expected net returns, and $\mathbf{z}_1$ second-order stochastically dominates $\mathbf{z}_2$. Show that, in that case, $\alpha_1^* > \alpha_2^*$.

6. **GRAD**: Two expected utility maximizing individuals utility functions u and v with $u', v' > 0 >$ and $u'', v'' < 0$, and have the same initial wealth $w > 0$. Each of them decides how much of his initial wealth to invest in a risky asset with gross return $\mathbf{z}$, with $\mathbb{E}[\mathbf{z}] > 1$ and $\mathbb{V}[\mathbf{z}] > 0$.

   Show that if the agent with utility function u is more risk averse than the one with utility function v, then the one with utility function u will invest less in the risky asset.