ECON 3810/CS5846/ECON 6676: HW2

Handed out: Mar. 2, 2017. Due: Mar. 14, 2017

1. Show that Sen's axioms α continues to hold if $\succ$ is a partial order, but show by example that Sen's β may fail to hold. That is, show that $C(\cdot, \succ)$ satisfies Sen's α if $\succ$ is a partial order (and not necessarily a preference order), although Sens β may not hold in this case.

2. GRAD: Let β' be the axiom that says if $x \in A - C(A)$ then there there exists a $y \in C(A)$ such that $x \notin C(\{x, y\})$. Show that Sen's α and axiom β' characterize choice functions C such that C is determined by a partial order.

3. (a) Suppose that you are an insurance broker, trying to decide whether to offer fire insurance to a business located in an old wooden structure with an oil furnace in the basement. What is a reasonable state space and outcome space for the decision? How do things change if you have reason to believe that the owner of the business was a convicted arsonist? (There's no "right" or "wrong" answer here. But you should argue for the reasonableness of your choices.)

   (b) Recall the "better red than dead" example from Prof. Halpern's notes. Define a reasonable state space that is independent of the acts "arm and "disarm". (The outcome space is still "dead", "red", "status quo", and "improved society".) Again, there is no unique "right" answer. But whatever answer you give, you should argue that the acts are independent of the states you choose.

4. Prove the following proposition from Prof. Halpern's notes:
   Proposition: Let f be an increasing function. Then maximin$(u) =$ maximin$(f(u))$ and maximax$(u) =$ maximax$(f(u))$. Show by example that $opt^{\alpha}(u)$ may not be the same as $opt^{\alpha}(f(u))$, and that $regret(u)$ may not be the same as $regret(f(u))$.

5. Prove the following proposition from Professor Halpern's notes:
   Proposition: Let f be a positive affine transformation. Then

   - maximin$(u) =$ maximin$(f(u))$

- $opt^\alpha(u) = opt^\alpha(f(u))$
- $regret(u) = regret(f(u))$.

6. Recall the algorithm for the location problem: The robot goes to $+1$, then -2, then $+4$, then -8, ..., $+2^{2n}$, -2^{2n+1}, until it finds the object.

   (a) Show that this algorithm has a competitive ratio of 9. This was done in class, so you just need to write the argument out carefully. That is, show that compared that, no matter where the object is located, it will take the robot at most 9 times as many steps to find the object using this algorithm as it would to go directly to the object. (Hint: compute exactly how long it will take the robot to get to an object at location $2^{2n}+m$ for $0 < m \leq 2^{2n+2}-2^{2n}$. Then do a a similar computation for an object at location $-2^{2n+1} - m$, for $0 \leq m < 2^{2n+3} - 2^{2n+1}$.)

   (b) Show that if $c < 9$, then for all k, there is a location N_k for the object such that the robot will take more than cN_k+k steps to find the object. (Note: this shows that this algorithm has competitve ratio no better than 9.)

7. Suppose that instead of increasing by powers of 2, we increase by powers of 3, so that the robot goes from +1 to -3 to 9, and so on.

   (a) Show that the competitive ratio is now 10, not 9.

   (b) GRAD: compute the competitive ratio if we increase by powers of x, for $x > 1$. Use calculus to show that the optimal choice is to take $x = 2$.

8. Show that $a \succ^3_{\mathcal{P}} a'$ implies $a \succ^4_{\mathcal{P}} a'$, where $\mathcal{P}$ is a set of probability distributions and $\succ^3_{\mathcal{P}}$ and $\succ^4_{\mathcal{P}}$ are the partial orders defined in Prof. Halpern's notes. (This result is also true if we replace $\succ^3$ by $\succeq^3$ and $\succ^4$ by $\succeq^4$, which is what the original version of the question asked. Proving either result is fine.)

9. Consider (a slight variant of) the ski rental problem from the notes. Suppose that you will be skiing for an unknown number of days, but at most N and at least one. You can either buy skis or rent them. You can rent skis for $\$r$/day, or buy them for $\$p$. Assume that whether or not you buy skis does not affect how many days you ski; that is, you will be skiing for d days out of N, independent of whether you buy or rent skis. (Note: this is almost certainly false for many people. Once they buy skis, they view it as a commitment and are less likely to quit skiing. The assumption makes sense only if you view someone

else as deciding whether or not you ski; your only decision is whether to buy or rent.) With this assumption, model the ski rental problem as a decision problem.

(a) Describe the states, acts, and outcomes for this problem.

(b) Which act(s) are optimal according to the maximin rule?

(c) Which act(s) are optimal acording to the minimax regret rule?

(d) Which act(s) give the optimal competitive ratio?

(e) Explain clearly why the assumption that your decision to buy or rent does not affect whether you ski is necessary in your analysis.

(f) **Bonus problem:** (not required to get full credit): Suppose that $p - r < Nr - p$. Show that randomization helps in terms of minimax regret. That is, show that there is a randomizaed act whose minimax regret is better than any deterministic act.

Note: the act(s) that are optimal depend on the values of the parameters p, r, and N. (Actually, all that matters is p/r and N.)