Experiment

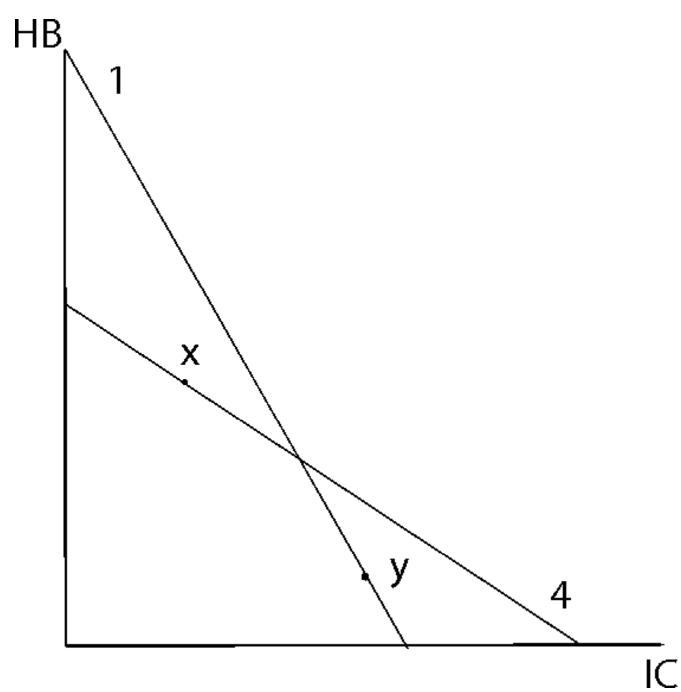
Did the subjects make choices "as if" they had a preference relation $\succ$ over bundles of (IC, HB)? If so, could we infer $\succ$ and predict future choices or offer advice about choices?

In situation 2 the amount of money was \$3.00 and the prices were $p_{HB} = .50$ and $p_{IC} = 1.00$; in situation 5 the amount of money was \$3.60 and the prices were $p_{HB} = .60$ and $p_{IC} = 1.20$. The affordable set was the same in these two cases. So if the framing of the question doesn't matter would expect the same choice in 2 as in 5.

22% of the subjects did not make the same choice in these two situations.

So we observe $x \succ y$ and $y \succ x$ for these people.

Lets look at situation 4 versus situation 1. In situation 4 the amount of money was \$4.20 and the prices were $p_{HB} = .80$ and $p_{IC} = 1.20$; in situation 1 the amount of money was \$3.60 and the prices were $p_{HB} = .40$ and $p_{IC} = 1.60$. The affordable sets in these two cases are graphed below.



If we observe choices x at 4 and y at 1 then we have $x \succ y$ and $y \succ x$. No one made choices like this.

Static Decision Theory Under Certainty

A set of objects X.

An individual is asked to express his preferences among these objects or is asked to make choices from subsets of X.

For $x, y \in X$ we can ask which, if either, is strictly preferred.

- If the individual says x is strictly better than y we write $x \succ y$, read as x is strictly preferred to y.
- $\succ$ is a binary relation on X.

Example 1: $X = \{a, b, p\}$, $b \succ a$, $a \succ p$ and $b \succ p$.

What if the answers also included $a \succ b$?

Axioms

Asymmetry: For any $x, y \in X$ if $x \succ y$ then not$[y \succ x]$.

Negative Transitivity: For any $x, y, z \in X$ if not$[x \succ y]$ and not$[y \succ z]$ then not$[x \succ z]$.

Proposition. The binary relation $\succ$ is negatively transitive iff $x \succ z$ implies that, for all $y \in X$, $x \succ y$ or $y \succ z$.

Example 2: $X = \{a, b, c\}$, $b \succ a$, $a \succ c$ and b ? c. If we have asymmetry and NT you also know how b and c must be ranked.

Definition. A binary relation $\succ$ is called a (strict) *preference relation* if it is asymmetric and negatively transitive.

Is Asymmetry a good normative or descriptive property? What about NT?

Weak Preference

Definition. For $x, y \in X$:

1. x is *weakly preferred* to y, $x \succeq y$, if not$[y \succ x]$.
2. x is *indifferent* to y, $x \sim y$, if not$[x \succ y]$ and not$[y \succ x]$.

Does the absence of strict preference in either direction require real indifference or could it permit non-comparability?

Example. $X = \{a, b, c\}$. Suppose a is not ranked (by $\succ$) relative to either b or c. If $\succ$ satisfies NT, then b and c are not ranked either.

An interesting alternative would be to ask about $\succ$ and $\sim$ separately. Then define $x \succeq y$ as either $x \succ y$ or $x \sim y$. This permits the possibility that x and y are not comparable.

Definition. The binary relation $\succeq$ on X is *complete* if for all $x, y \in X$, $x \succeq y$, $y \succeq x$ or both. It is *transitive* if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Proposition. Let $\succ$ be a binary relation on X.

1. $\succ$ is asymmetric iff $\succeq$ is complete.
2. $\succ$ is negatively transitive iff $\succeq$ is transitive.

Proof of $\Rightarrow$

1. By asymmetry of $\succ$ there is no pair $x, y \in X$ such that both $x \succ y$ and $y \succ x$. So at least one of not$[x \succ y]$ and not$[y \succ x]$ is true. Thus for any $x, y \in X$ either $y \succeq x$ or $x \succeq y$ or both. This is completeness.
2. Using the definition of $\succeq$, negative transitivity of $\succ$ is: for any $x, y, z \in X$, $y \succeq x$ and $z \succeq y$ implies $z \succeq x$. This is transitivity.

$\Leftarrow$ will be on homework 1.

Transitivity

Why do we care about transitivity?

Remark: If $\succ$ is a preference relation then $\succ$ is transitive.

Normative property?

Important for choice.

Example. $X = \{a, b, p\}$. Consider a sequence of choices from among pairs.

1. $\{a, b\}$, $a \succ b$ and a is chosen.
2. $\{a, p\}$, $p \succ a$ and p is chosen.
3. $\{p, b\}$, $b \succ p$ and b is chosen.
4. $\{a, b\}$ ...

Without transitivity can get cycles.

Remark: If $\succ$ is a preference relation then $\succ$ is acyclic, i.e. $[x_1 \succ x_2 \succ \ldots x_{n-1} \succ x_n] \Rightarrow [x_1 \neq x_n]$.

Choice

Extend binary comparisons to choice over a set of more objects.

A *finite* set of objects X. Let $P(X)$ be the set of all non-empty subsets of X.

Definition. For $\succ$ a preference relation on X define $c(\cdot, \succ)$ by, for $A \in P(X)$,

$$c(A, \succ) = \{x \in A : \text{for all } y \in A, y \not\succ x\}.$$

Interpretation: $c(A, \succ)$ is the set of alternatives chosen from A by a decision maker with preferences $\succ$.

Remark: If $x, y \in c(A, \succ)$ then $x \sim y$.

Proposition. For $\succ$ a preference relation on a finite set X,

$$c(\cdot, \succ) : P(X) \to P(X).$$

What else do we know about $c(\cdot, A)$?

Consider general choice functions and ask what is special about $c(\cdot, A)$.

Definition. A *choice function* for X is a function $c : P(X) \to P(X)$ such that for all $A \in P(X)$, $c(A) \subset A$.

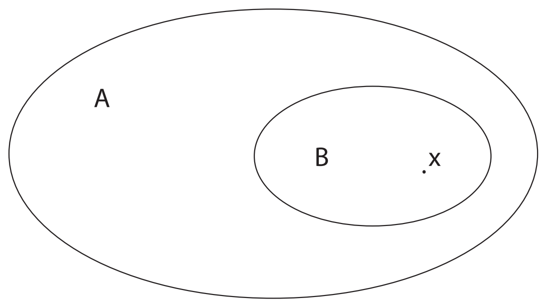
Clearly, $c(\cdot, \succ)$ is a choice function.

Can any choice function be generated by some preference relation $\succ$? No.

Example. $X = \{a, b, c\}$.

1. $c(\{a, b, c\}) = \{a\}$ and $c(\{a, b\}) = \{b\} \Rightarrow$ a violation of asymmetry.
2. $c(\{a, b\}) = \{a, b\}$ and $c(\{a, b, c\}) = \{b\} \Rightarrow$ a violation of NT.

Axioms

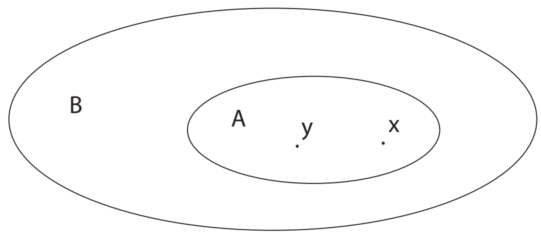


Sen's α. If $x \in B \subset A$ and $x \in C(A)$, then $x \in C(B)$.

Independence of Irrelevant Alternatives.

Proposition. If $\succ$ is a preference relation then $c(\cdot, \succ)$ satisfies Sen's α.

Proof. Suppose there are sets $A, B \in P(X)$ with $B \subset A$, $x \in c(A, \succ)$ and $x \notin c(B, \succ)$. Then there is a $y \in B$ such that $y \succ x$. Since $B \subset A$ we have $y \in A$ and $y \succ x$. Thus $x \notin c(A, \succ)$. A contradiction.



Sen's β. If $x, y \in c(A), A \subset B$ and $y \in c(B)$ then $x \in C(B)$.

Proposition. If $\succ$ is a preference relation then $c(\cdot, \succ)$ satisfies Sen's β.

Proof. Since $x \in c(A, \succ)$ and $y \in A$ we have $y \not\succ x$. By definition, $y \in c(B, \succ)$ implies that for all $z \in B, z \not\succ y$. By negative transitivity, $y \not\succ x$ and $z \not\succ y$ implies $z \not\succ x$. Since $x \in B$ and this holds for all $z \in B$ we have $x \in c(B, \succ)$.

Are there any other restrictions on $c(\cdot, \succ)$ that follow from $\succ$ being a preference relation? No.

Proposition. If a choice function c satisfies Sen's α and β, then there is a preference relation $\succ$ such that $c(\cdot) = c(\cdot, \succ)$.

Define the "revealed preference" relation $\succ$ by

$$x \succ y \text{ if } x \neq y \text{ and } c(\{x, y\}) = \{x\}.$$

To prove the proposition we need to show that $\succ$ is a preference relation and that $c(\cdot) = c(\cdot, \succ)$.

Proof

To show that $\succ$ is a preference relation we need to show that it is asymmetric and negatively transitive.

1. Asymmetry. Suppose for some x and y, that $x \succ y$ and $y \succ x$. Then $c(\{x, y\}) = \{x\}$ and $c(\{x, y\}) = \{y\}$. A contradiction.

2. Negative Transitivity. Suppose that for some $x, y, z \in X$ we have $z \not\succ y$ and $y \not\succ x$. We need to show that $z \not\succ x$. This is $x \in c(\{x, z\})$. By Sen's α, showing that $x \in c(\{x, y, z\})$ is sufficient. Suppose $x \notin c(\{x, y, z\})$. Then at least one of y and z are in $c(\{x, y, z\})$.

   Suppose $y \in c(\{x, y, z\})$. Then by Sen's α, $y \in c(\{x, y\})$. By $y \not\succ x$ we have $x \in c(\{x, y\})$. By Sen's β this implies that $x \in c(\{x, y, z\})$.

   Suppose that $z \in c(\{x, y, z\})$. Then by Sen's α, $z \in c(\{y, z\})$. By $z \not\succ y$ we have $y \in c(\{y, z\})$. By Sens' β this implies that $y \in c(\{x, y, z\})$. By the previous argument this implies that $x \in c(\{x, y, z\})$.

We also need to show that for each $A \in P(X)$, $c(A) = c(A, \succ)$.

1. Suppose $x \in c(A)$. Then by Sen's α, $x \in c(\{x, y\})$ for all $y \in A$. Thus for all $y \in A$, $y \not\succ x$. So $x \in c(A, \succ)$.

2. Suppose $x \in c(A, \succ)$. Then for all $y \in A$, $y \not\succ x$. So for all $y \in A$, $x \in c(\{x, y\})$. Suppose $x \notin c(A)$. Then there is some $z \in A$, $z \neq x$ such that $z \in c(A)$. By Sen's α, $z \in c(\{x, z\})$. Then $c(\{x, z\}) = \{x, z\}$, $\{x, z\} \subset A$ and $z \in c(A)$. So by Sen's β, $x \in c(A)$. A contradiction.

So we know,

$$[\text{Sen's } \alpha \text{ and } \beta \text{ for } c(\cdot)] \Leftrightarrow$$

$$[c(\cdot) = c(\cdot, \succ) \text{ for the preference relation } \succ]$$

WARP

There is an alternative equivalent way to state Sen's α and β.

This is Houthaker's Axiom which is also called the *Weak Axiom of Revealed Preference (WARP)*.

WARP: If x and y are both in A and B and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ and $y \in C(A)$.

Proposition. $c(\cdot)$ satisfies Sen's α and β if and only if it satisfies WARP.

Partial Orders

Completeness of $\succeq$ is questionable from both a descriptive and a normative point of view.

Definition. $\succ$ is a *partial order* if it is an asymmetric and transitive binary relation.

We can define a choice function as before. What properties does it have?

Sen's α still holds, but Sen's β may fail. (On homework 1.)

Now we would not want to define $\sim$ as before. $x \not\succ y$ and $y \not\succ x$ could express indifference or non-comparability.

An alternative approach is to include a positive expression of indifference, i.e. preferences described by the pair $(\succ, \sim)$.