Ordinal Representations

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Note: The marked exercises are additional problems I thought would be amusing to think about.

1 Theories of Measurement

The representation of preference orders by numerical scales is an important part, but only part, of the theory of measurement. It is worthwhile to understand what we mean by *measurement* before we begin to think about how to measure preferences on a numerical scale.

Measurement theory is about assigning numbers to things. Think degrees Fahrenheit, pitch, the Richter scale, the category of a hurricane, batting average and the runs-created index, quarterback efficiency, ... These are all attempts to scale different kinds of things. Theories of measurement help us to keep straight what we can and cannot learn from numerical representations. For instance, the statement I once heard on TV, that "tomorrow will be twice as hot as today" because today's high was in the low 20s and tomorrow's was expected to be in the mid 40s is meaningless. (Why?) Not all attributes of numbers are relevant to all measurement systems.

Most generally, a measurement system is One way (not the best way, I think, and not standard) of describing measurement systems is to describe the transformations of the measurement scalethat we expect to preserve relationships. Here is a rough categorization of measurement systems:

type	description	example
nominal	equality, $1-1$ transformations	"1 if vaccinated, 0 if not"
ordinal	order relation, strictly increasing transformation	preference
interval	difference matters, positive affine transformations	degrees C and F
ratio	difference and ratios matter positive linear transformations	length in inches, degrees K
absolute	all properties matter identity transformation	probability

A better way to describe measurement systems is to describe the universe of objects being measured, a collection of relations among these objects, a corresponding collection of relations among the real numbers, and a map from objects to reals that preserves the relationships. We will not pursue this here. A good book on the topic is Louis Narens' *Abstract Measurement Theory*

2 *Ordinal* and *Cardinal* Representations

Briefly, in the scheme above, interval, ratio and absolute measurement systems are all *cardinal* measures, while utility, with which we measure preferences, is *ordinal*.

We are given a *preference order* $\succ$ on X, which wwe want to scale to the real numbers. The map from objects to numbers, which preserves order properties, is called a *utility function*.

Definition 1. *A* utility representation *of the preference order* $\succ$ *is a function* $U : X \to \mathbf{R}$ *such that* $x \succ y$ *if and only if* $u(x) > u(y)$.

What do we mean by an ordinal representation? First, a representation is a numerical scaling — a thermometer to measure preference. Thus if x is better than y, x gets a higher utility number than y, just as if New York City is hotter than Boston, NY gets a higher temperature number. But with utility, only the ordinal ranking matters. Temperature is not an ordinal scale. New York is only slightly hotter than Boston, while Miami is much hotter than Cleveland.

$$T(\text{Miami}) - T(\text{Cleveland}) > T(\text{New York}) - T(\text{Boston}) > 0$$

The temperature difference between New York and Boston is smaller than the temperature difference between Miami and Cleveland. But to say that

$$u(x) - u(y) > u(a) - u(b) > 0$$

does not mean that the incremental satisfaction from x over y is more than the incremental satisfaction from a over b. We express this as follows:

Definition 2. *A utility representation for $\succ$ is an* ordinal *measure of preference. That is, if U is a utility representation for $\succ$ and $f : \mathbf{R} \to \mathbf{R}$ is a strictly increasing function, $f \circ U$ is also a utility representation for $\succ$.*

3 Why do we want to measure preferences?

Summary: An ordering is just a list of pairs, which is hard to grasp. A utility function is a convenient way of summarizing properties of the order. For instance, with expected utility preferences of the form $U(p) = \sum_a u(a)p_a$, risk aversion — not preferring a gamble to its expected value — is equivalent to the concavity of u. The curvature of u measures how risk-averse the decision-maker is.

Optimization: We want to find optimal elements of orders on feasible sets. Sometimes these are more easily computed with utility functions. For instance, if U is C^1 and B is of the form $\{x : F(x) \leq 0\}$, then optima can be found with the calculus.

So why not start with utilities?

- Preferences, after all, are the primitive concept, and we don't know that utility representations exist for all kinds of preferences we'd want to talk about.
- Some characteristic properties of classes of preferences are better understood by expressing them in terms of orderings.
- Preferences are the primitive concept, and some properties of utility functions are not readily interpreted in terms of the preference order.

4 When do ordinal representations exist?

There are really two questions to ask:

- Does every preference order have a representation? More generally, what binary relations have numerical representations?
- Does every function from X to $\mathbf{R}$ represent some preference order? That is, does every function from X to $\mathbf{R}$ describe some preference relation?

The second question is easy. For a given $U : X \to \mathbf{R}$, define $x \succ_U y$ iff $U(x) > U(y)$.

Theorem 1. *For any domain X and function $U : X \to \mathbf{R}$, the binary relation $\succ_U$ is a preference order.*

Proof. Asymmetry is obvious. If $x \succ_U y$, then $U(x) > U(y)$ and so not $U(y) > U(x)$, so not $y \succ_U c$. To check negative transitivity, suppose that not $x \succ_U y$ and not $y \succ z$. Then $U(x) \geq U(y)$ and $U(y) \geq U(z)$, so $U(x) \geq U(z)$, so not $z \succ_U y$. □

The answer to the first question depends on the cardinality of X and the properties of $\succ$. Recall that an asymmetric relation $\succ$ is a

partial order: if it is *transitive*;

preference order: if it is negatively transitive;

We now describe several cases.

4.1 Finite X

This is an intermediate case — theorem 3 covers this case as well, but finiteness makes clear what's going on.

Theorem 2. *Suppose X is finite. If $\succ$ is a preference order, then it has a utility representation.*

Recall K. Proposition 2.3; in particular, if $\succ$ is a preference relation, it is transitive and irreflexive. Also recall K. Proposition 2.4*d*: If $w \succ x$, $x \sim y$, and $y \succ z$, then $w \succ y$ and $x \succ z$.

Proof. Define $W(x) = \{y : x \succ y\}$. Define $U(x) = \#W(x)$.

1. $U(x)$ is well-defined. That is, it exists for every x.
2. If $x \succ y$, then $U(x) > U(y)$. If $z \in W(y)$, then $z \in W(x)$, so $\#W(x) \geq \#W(y)$. Furthermore, $y \notin W(y)$ but $y \in W(x)$, so $\#W(x) > \#W(y)$, that is, $U(x) > U(y)$.
3. If $U(x) > U(y)$, then $x \succ y$. Observe first that we cannot have $y \succ x$, since otherwise $U(y) > U(x)$, which is a contradiction. If $x \not\succ y$, then $x \sim y$. But this cannot happen either. If $z \in W(x)$, then by 2.4d, $z \in W(y)$, and vice versa, so if $x \sim y$, then $W(x) = W(y)$, and so $U(x) = U(y)$, which is a contradiction.

□

4.2 Denumerable X

4.2.1 Preference orders

Preferences on countable sets can be more complicated. $x_1 \prec x_2 \prec \cdots$ has no maximal element. $x_1 \succ x_2 \succ \cdots$ has no minimal element. If $x_1 \succ x_2$, $x_{2k+1} \succ x_{2k-1}$ and $x_{2k} \prec x_{2k-2}$, then there is neither a maximal nor a minimal element.

Why won't the construction of Theorem 2 work? Nonetheless, every preference order has a representation.

Theorem 3. *Suppose X is denumerable. If $\succ$ is a preference order, then it has a utility representation.*

Proof. We will make use of K. Proposition 2.4.d — in particular, if $x \sim y$ and $y \succ z$, then $x \succ z$. The art of the proof is to define a candidate utility function and see that it works.

Begin by indexing X: $X = \{x_1, x_2, \ldots\}$, and consider a preference order $\succ$. For each $x \in X$ define $W(x) = \{y : x \succ y\}$, the "worse than x" set. Define $N(x) = \{n : x_n \in W(x)\}$; the set of indices of elements in the worse than x set. Finally, define

$$U(x) = 0 + \sum_{n \in N(x)} \left(\frac{1}{2}\right)^n$$

We must show that U is a utility representation for $\succ$; that is, $U(x) > U(y)$ if and only if $x \succ y$.

Suppose that $x \succ y$. Since $\succ$ is transitive and irreflexive, $W(y) \subsetneq$

$W(x)$. Consequently $N(y) \subsetneq N(x)$, and so

$$\begin{aligned} U(x) &= 0 + \sum_{x \in N(x)} \left(\frac{1}{2}\right)^n \\ &= 0 + \sum_{x \in N(y)} \left(\frac{1}{2}\right)^n + \sum_{x \in N(x)/N(y)} \left(\frac{1}{2}\right)^n \\ &> 0 + \sum_{x \in N(y)} \left(\frac{1}{2}\right)^n \\ &= U(y). \end{aligned}$$

Suppose that $U(x) > U(y)$. There are only three possibilities for the order of x and y: $x \succ y$, $x \sim y$ and $y \succ x$. We will rule out the last two. The third is ruled out, because we have already shown that $y \succ x$ implies $U(y) > U(x)$. Suppose $x \sim y$. If $z \in W(y)$, then 2.4.d implies that $z \in W(x)$ and vice versa. Thus $N(x) = N(y)$ and so $U(x) = U(y)$. The only remaining possibility is $x \succ y$. □

4.2.2 Partial orders

Indifference need not be transitive in a partial order, so there is no possibility of getting a full numerical representation. In the following figure, if there is a path in the direction of the arrows from x to y, then $x \succ y$. Any binary relation with such a representation must be transitive since if there is a path from a to b and a path from b to c, conjoining the two paths gives a path from a to c. The relation will be asymmetric if and only if there are no loops, thatis, no paths that start from some vertex a and return to a. In this figure, $a \sim b$, $b \sim c$ and $a \succ c$. If $\succ$ had an ordinal representation U, then it would follow that $U(a) = U(b)$, $U(b) = U(c)$, and $U(a) > U(c)$, which is impossible. However, it has a representation in the following weaker sense:

Definition 3. *A* weak *or* one-way utility representation *of the partial order* $\succ$ *is a function* $U : X \to \mathbf{R}$ *such that if* $x \succ y$*, then* $U(x) > U(y)$*.*

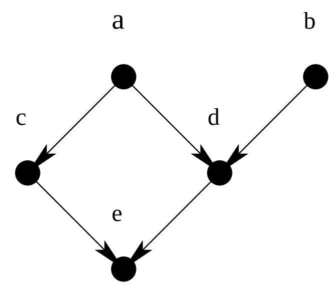

Figure 1: A Partial Order.

A one-way representation for the partial order $\succ$ in Figure 1 is $U(e) = 0$, $U(c) = 1$, $U(d) = 2$, $U(a) = 3$ and $U(b) = 4$. Another one-way representation is $V(e) = 0$, $V(c) = 2$, $V(d) = 1$, $V(a) = 3$ and $V(b) = 2$.

Theorem 4. *Suppose X is denumerable. If $\succ$ is a partial order, then it has a weak utility representation.*

Proof. The same construction as that in the proof of Theorem 3 works here. Try it yourself. □

If $\succ$ is a partial order on a finite set X, then $C(B, \succ)$ exists for all $B \in P^+(X)$, and if $x \in B$ maximizes U on B, then $x \in C(B, \succ)$. However the converse is false. For instance, with the representation U for the $\succ$ of Figure 1, only b maximizes utility on $\{a, b, c, d, e\}$ but $C(\{a, b, c, d, e\}, \succ) = \{a, b\}$. With the representation V, only a maximizes utility on $\{a, b, c, d, e\}$. If there is a function W that "gets it right" on every subset, then in particular it would get it right on every pair, and so $\succ = \succ_U$. Thus $\succ$ would have to be a preference order, which it evidently is not.

Exercise 1. *Which of Sen's axioms α and β fail to hold? Find axioms which characterize those $C(B)$ which are a $C(B, \succ)$ for some partial order $\succ$.*

Exercise 2. *Let $\succ$ be a partial order on a denumerable set X. Define $\succeq$ and $\sim$ in the usual way. Define $x \approx y$ if for all z, $x \sim z$ iff $y \sim z$. Show that*

1. *$\approx$ is an equivalence relation.*

2. *If $w \approx x$, $x \succ y$, and $y \approx z$, then $w \succ y$ and $x \succ z$.*

3. There is a function $U : X \to \mathbf{R}$ such that if $x \succ y$, then $U(x) > U(y)$ and $x \approx y$ iff $U(x) = U(y)$.

Does this still hold true if $\succ$ is only acyclic rather than transitive?

Alternative representation strategies are possible. One such strategy is motivated by the *Pareto order*. This notion comes from economics, and is a way of ranking social situations. Imagine an apartment with three roommates. They must decide on which of some large number of days to have a party. The set of all possible dates is X. Each roommate has a preference order on X. Number the roommates 1 through 3 and let $\succ_i$ denote roommate i's preference order. The *(strong) Pareto order* $\succ$ on X is defined by saying that $x \succ y$ if and only if $x \succ_i y$ for all i. That is, $x \succ y$ if all roommates agree that x is a better date than y.[1]

The Pareto order is a partial order; it is easily seen to be transitive and symmetric. It may not be a preference order — negative transitivity may fail.

Exercise 3. *Construct an example to show how negative transitivity may fail for the Pareto order.*

Although the Pareto order is only a partial order, it has a kind of numerical representation. Each roommate has a preference order, and so for each roommate i there is a utility function U_i such that $x \succ_i y$ iff $U_i(x) > U_i(y)$. It follows, then, that $x \succ y$ iff for all i, $U_i(x) > U_i(y)$. In other words, we can represent the Pareto partial order by checking three utility functions, and if x beats y on all three scales, then $x \succ y$. If the set X of dates is large, this *multiple-utility* representation can still provide a description of the partial order $\succ$ which is more parsimonious than simply listing all the pairs or ordered dates.

The nice fact is that this idea works in general. Partial orders have multiple-utility representations. Whether a particular multiple-utility representation is useful or not depends upon how many utility functions are

[1]Afficionados will notice that what I have actually defined is the *strong* Pareto order. The regular Pareto order would require that $x \succ y$ iff there is an i for which $x \succ_i y$ and for no j is $y \succ_j x$, that is, someone prefers x to y and no one else objects.

needed for a representation, but oftentimes partial orders on large sets can be described by a very few functions.

Definition 4. *A* multiple-utility representation *for the partial order $\succ$ on a set X of alternatives is a set $\mathcal{U}$ of functions $U : X \to \mathbf{R}$ such that $x \succ y$ iff $U(x) > U(y)$ for all $U \in \mathcal{U}$.*

The pair of utility functions $\{U, V\}$ is a multiple utility representation for the partial order $\succ$ of Figure 1. The two functions disagree on the order of the pairs (a, b), (b, c) and (c, d), and these are precisely the pairs that are not ranked by $\succ$.

Theorem 5. *A binary relation $\succ$ on X has a multiple-utility representation if and only if it is a partial order.*

We did not prove this theorem in class, and you are not responsible for it, so the next few paragraphs are only for fun.

The "only if" direction is obvious (but make sure you agree), so I will prove here only the "if" direction. The rest of this section is devoted to the proof. The key idea is that of an *extension* of a binary relation. Suppose that the set $\mathcal{U}$ is a multiple utility representation for $\succ$. For each $U \in \mathcal{U}$, $\succ_U$ is a preference order, and $x \succ y$ iff $x \succ_U y$ for all $U \in \mathcal{U}$. Each $\succ_U$ is an *extension* of $\succ$ to a preference relation: It agrees with $\succ$ whenever $\succ$ makes a comparison, and adds enough additional rankings to make a preference order. This suggests a proof strategy: Let $\mathcal{U}$ denote the set of utility functions of *all* extensions of $\succ$ to a preference order. Perhaps this set will do the trick. If it does, it may well not be the smallest set which represents $\succ$. We saw this in class. The $\succ$ of Figure 1 can be represented by only two utility functions, but it has 11 distinct extensions. But this is another issue. Now we formalize this proof idea.

Definition 5. *A binary relation $\succ'$ on X* extends *the binary relation $\succ$ on X if and only if $x \succ y$ implies that $x \succ' y$.*

So an extension of $\succ$ will have all the comparisons that $\succ$ does, and perhaps more.

Every partial order has an extension which is a preference order. Suppose U is a weak representation for $\succ$ on X. Then the binary relation $\succ_U$ is an extension of $\succ$, and it is a preference order (because, after all, it has a utility representation).

Let $\mathcal{E}$ denote the set of all preference orders which extend $\succ$, and let $\mathcal{U}$ denote a set of functions with the property that for each $\succ' \in \mathcal{E}$ there is a utility representation $U \in \mathcal{U}$. We have just shown that $\mathcal{E}$ and $\mathcal{U}$ are non-empty.

If $x \succ y$, then $x \succ' y$ for every extension of $\succ$. Thus $U(x) > U(y)$ for all $U \in \mathcal{U}$. We need to show the converse, that if $U(x) > U(y)$ for all $U \in \mathcal{U}$, then $x \succ y$. Equivalently, and this is key, if $x \not\succ y$, then there is a $U \in \mathcal{U}$ such that $U(y) \geq U(x)$.

If $x \not\succ y$, then either $y \succ x$ or y and x are not compared by $\succ$. If $y \succ x$, we have already seen that $U(y) > U(x)$ for all $U \in \mathcal{U}$. The remaining case is where x and y are unranked by $\succ$. In this case we need to show that there are a U' in $\mathcal{U}$ such that $U'(y) \geq U'(x)$.

Suppose, then, that x and y are unranked by $\succ$. Extend $\succ$ to a new partial order $\succ'$ by adding the comparison of x and y. That is, define $\succ'$ as follows: (1) List all the comparisons made by $\succ$. (2) Add $y \succ' x$. (3) Add $a \succ' b$ if there is a chain of elements $a_0, a_1, \ldots, a_n$ where $a_0 = a$, $a_n = b$, and for all other a_i, either $a_i \succ a_{i+1}$ or $a_i = y$ and $a_{i+1} = x$.[2]

In general, the extension $\succ'$ will not be negatively transitive, but it will be a partial order. And showing this proves the theorem. Why? Because if it is, then it has a weak representation U'. Since $\succ'$ is an extension of $\succ$, U' is a weak representation for $\succ$, and hence U' is in $\mathcal{E}$. And since U' is a weak representation of $\succ'$, $U'(y) > U'(x)$.

Finally, then, why does adding "y is better than x" to $\succ$ and closing it by transitivity to make $\succ'$ guarantee that $\succ'$ is a partial order? Clearly $\succ'$ is transitive, because we added all relations that could be derived by transitivity. We need to show that it is asymmetric. If it were the case that

[2] We say that $\succ'$ is the *transitive closure* of the relation formed by starting with $\succ$ and adding to it the ordered pair (y, x).

for some a and b, $a \succ' b$ and $b \succ' a$, transitivity implies that $a \succ' a$. That is, $\succ'$ would not be reflexive. So we need only show that $\succ'$ is reflexive.

Suppose $\succ'$ it is not reflexive. Then there is some element a such that $a \succ' a$. This comparison was not present in the original $\succ$, and so it must have been added by step (3) in the construction of $\succ'$. So there is a chain of elements $a_0, \ldots, a_n$ such that $a = a_0 = a_n$, and for all i, $a_i \succ' a_{i+1}$. That is,

$$a \succ' a_2 \succ' \cdots \succ' a_{n-1} \succ' a$$

Furthermore, one of the comparisons must involve an $a_i = y$ and an $a_{i+1} = x$. Why? because all of the other comparisons are already in $\succ$, so if this were false, we could conclude from the transitivity of $\succ$ that $a \succ a$, which contradicts asymmetry.

Since this is a preference cycle, we can enumerate it starting from x:

$$x \succ a_{i+2} \succ a_{i+3} \succ \cdots \succ a_{n-1} \succ a \succ a_1 \cdots \succ a_{i-1} \succ y.$$

We can write $\succ$ rather than $\succ'$ because all of the comparisons of the cycle other that $y \succ' x$ were already in $\succ$. But since $\succ$ is a partial order, it is transitive, and so this implies that $x \succ y$. This contradicts our hypothesis that $\succ$ did not compare x and y, and this completes the proof.

4.3 Uncountable X

Not all preference orders are representable.

Example:
Let $X = \mathbf{R}^{\mathbf{2}}_{+}$. Define the relation $(x_1, x_2) \succ (y_1, y_2)$ iff $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. It is called the *lexicographic order* on $\mathbf{R}^{\mathbf{2}}$. In Figure 1, better points are to the right, but if two points are equally far to the right, the top point is better. This order has no utility representation. To see why, choose two distinct points on each vertical line. Suppose there were a utility representation U. The top point t_x on the line with first coordinate x must map to a higher number than the bottom point b_x on that line. Now consider

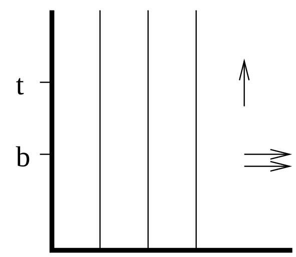

Figure 2: The Lexicographic Order.

the collection of intervals $\{[U(b_x), U(t_x)] : x \geq 0\}$. These intervals are all disjoint. Furthermore, since they are non-degenerate, each contains a rational number. These rational numbers are all distinct, and we have one for each vertical line, so if a utility function exists, there must exist an uncountable collection of rational numbers. No such collection exists; the rationals are countable. So U must not in fact exist. □

Exercise 4. *Show that the lexicographic order is in fact a preference order.*

4.3.1 Existence of ordinal representations

Again, this subsection is just for kicks. It complements the discussion in Kreps, and you are not responsible for it's contents.

Another example will illustrate what an ordering that has an ordinal representation looks like.

Example:
Take X to be $\mathbf{R}^{\mathbf{2}}_{+}$. For each $x \in X$, define $l(x)$ to be the line with slope -1 through x intersected with X. Define $x \succ y$ if y lies above the line $l(x)$. The situation is illustrated in figure 3. Point y is preferred to point x because y lies above $l(x)$. It is easy to see that $\succ$ is a preference order. It is also easy to see that $y \sim x$ if and only if $y \in l(x)$. The lines with slope -1 are called *indifference curves*, since two points on the same line are indifferent to each other. Ordering the points comes down to ordering the indifference

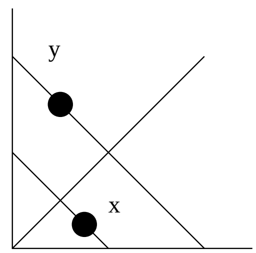

Figure 3: A representable order.

curves. Lines farther out are better, so a natural utility representation is to measure how far each line is from the origin; that is, where it intersects the diagonal. □

For a utility representation to exist, the order $\succ$ on X must "look like" the $>$ order on the real line. The order $\geq$ is complete, transitive and reflexive, and so is $\succeq$ for any preference order $\succ$. The $\geq$ order on $\mathbf{R}$ has another property that, strictly speaking, has to do with the structure of $\mathbf{R}$ as well as the order. The rational numbers $\mathbf{Q}$ are a countable subset of $\mathbf{R}$ with the property that if a, b are in $\mathbf{R}/\mathbf{Q}$ and $a > b$, then there is a rational number $r \in \mathbf{Q}$ such that $a > r > b$. It is exactly this property that fails in the lexicographic example.

Definition 6. *A set $Z \subset X$ is* order-dense *if and only if for each pair of elements $x, y \in X/Z$ such that $x \succ y$ there is a $z \in Z$ such that $x \succ z \succ y$.*

Theorem 6. *For a preference order $\succ$ on X, a utility representation exists if and only if X contains a countable order-dense subset.*

Proof sketch: Essentially the denumerable construction works: Let Z denote the countable order-dense set, and let $N(x)$ denote the set of indices of elements of Z that are worse than x. Proceed as before. □

The existence of a countable order-dense set is an example of an *Archimedean assumption*. It is required so that the preference order "fits in" to $\mathbf{R}$. The set $\mathbf{R}$ is an example of an *ordered field*. The rational numbers are another example. There are also ordered fields that strictly contain $\mathbf{R}$ —

the so-called hyperreal or non-standard numbers. One can show that if $\succ$ is any preference relation, it can be represented in some ordered field. If X is uncountable, it certainly cannot be represented in $\mathbf{Q}$, and in order to fit into $\mathbf{R}$, it must be "small enough". This is what order-denseness does.

Exercise 5. *State and prove a representation theorem for partial orders on a non-denumerable X.*

Clearly lexicographic preferences have no countable order-dense set, since any order-dense set must contain at least one element on each vertical line, and there are an uncountable number of such lines. The points in $\mathbf{R}^{\mathbf{2}}_{+}$ with rational coordinates are order-dense for $\succ$ in the second example.

4.3.2 Continuous representations

The point of choice theory is to describe choice behavior by deriving the choice functions $C(B, \succ)$. When X is finite, or each B we care about is finite, the fact that $\succ$ is a preference order is enough to derive that $C(B, \succ) \neq \emptyset$. When B is not finite, choice functions may be empty.

Example:
X is the set of non-negative integers. $x \succ y$ iff $x > y$. B is the set of even integers. □

So we want to find restrictions on $\succ$ and on the set of admissible $\mathcal{B}$ of feasible sets B such that $C(B, \succ) \neq \emptyset$ for all all $B \in \mathcal{B}$. For example, if X is denumerable and $\mathcal{B}$ is taken to be the collection of all non-empty finite subsets of X, K. Proposition 2.8 still holds: If $\succ$ is a preference, then $C(B, \succ) \neq \emptyset$.

When X is not denumerable, more assumptions are needed. The setting that comes up most often in modelling applications has X a closed subset of a Euclidean space. If $\succ$ has a utility representation, then

$$C(B, \succ) = \text{argmax}\{U(x), x \in B\}$$

We would like to know conditions on U and B that will guarantee the existence of solutions to this problem.

A natural generalization of finiteness to this setting is *compactness.*

Definition 7. *A set B in $\mathbf{R^n}$ is* compact *iff it is both closed and bounded.*

A basic fact of real analysis is **Weierstrass' Theorem:** Every continuous function has a maximum on every compact set. Formally, if U is continuous and B is compact, then there is an $x \in B$ such that for all $y \in B$, $U(x) \geq U(y)$. So if we're willing to accept the restriction that $\mathcal{B}$ contains only compact sets, then a sufficient condition guaranteeing choice is that $\succ$ have a continuous utility representation. What conditions on $\succ$ guarantee that it has a continuous utility representation?

Recall that a preference order is just a set of pairs of alternatives: $\{(x, y) \in X \times X : x \succ y\}$.

Definition 8. *A preference order $\succ$ is* continuous *iff $\{(x, y) \in X \times X : x \succ y\}$ is open in $X \times X$.*

Theorem 7. *A preference order has a continuous utility representation iff it is continuous.*

Proof. See Debreu (1954). A cleaner discussion can be found in Rader (1963). □

Exercise 6. *Show that if $\succ$ is open, the sets $W(x)$ and the corresponding "better than" sets $B(x) = \{y : y \succ x\}$ are open for all $x \in X$. Is the converse true?*

5 Characterizing preferences through their representations

Another aspect of representation theory is the characterization of preferences with certain kinds of representations.

Example:
For instance, consider choice under uncertainty. Suppose there are a finite set of rewards $R = \{r_1, \ldots r_n\}$. A *lottery* is a probability distribution on rewards; that is, a vector $(p_1, \ldots, p_n)$. Decision makers have preferences on lotteries. A utility function U on lotteries is an *expected utility* representation if there is a function $u : R \to \mathbf{R}$ such that

$$U(p_1, \ldots, p_n) = p_1 u(r_1) + \cdots + p_n u(r_n)$$

We would like to characterize or otherwise identify those preference orders that have an expected utility representation. □

This is just one example of how we might like to identify a class of preferences based on properties of a numerical representation. Another example, which sits apart from choice under uncertainty, follows.

5.1 Additive Separability

The theory presented so far treats objects of choice as primitive abstract entities. But in real choice problems the objects of choice have structure, and this structure may suggest meaningful restrictions on preferences. Here I want to think of objects of choice as bundles of attributes. The classic example of this is the *commodity bundle* in economic analysis. When I go to the grocery store I don't just choose coffee or tea. I also have to choose lemon or sugar, milk or cream, etc. If the store has no fresh lemons, I may choose to put coffee rather than tea into my shopping basket. At a good restaurant one puts together an entire meal from a list of appetizers, first courses, entrees and desserts. One chooses the meal, but each possible meal is described by a list of these attributes. Choice under uncertainty offers another example of this phenomenon, which will be discussed at the end of this section.

How much utility do I get from a box of Kellogg's Corn Flakes? It is hard to answer this question because how much I like my corn flakes depends upon whether we have milk in the fridge, and what bugs are living in the sugar bowl. I never consume cereal alone, but only as part of a breakfast meal. I have to consider all of the attributes together, and for breakfast I cannot value one attribute independently of the others. Nonetheless one can imagine situations where it may be sensible to value each object independently. Suppose you are buying health insurance. You can describe the policy by listing all of the possible health events that could happen to you, and the net payout from the policy in each event. Thus a policy is just a list of attributes. Here it is plausible that you could talk meaningfully of the value of the surgical coverage, or the value of the prescription drug coverage. That is, one can talk meaningfully about preferences over each attribute, and think about aggregating them to get aggregate preferences over policies.

In formalizing this idea, objects of choice may be thought of as bundles of attributes. Cars may be characterized by gas milage, engine power, quality of the ride, etc. Utility of a given car depends upon the whole bundle of characteristics, but if the characteristics are independent, we may be able to sensibly ask after the value of gas milage, and so forth. When we can, utility is said to be *additive* in the attributes. The general question is, when objects of choice can be described by a collection of factors, when can one define utility on each factor, and when is utility of choice objects additive in the utilities of the factors. Expected utility is a particular example of this, but far from the only example.

Suppose that X is a product space: $X = X_1 \times \cdots \times X_n$. Each $x \in X$ is a bundle of attributes or characteristics. Each X_i is a *factor*. Suppose for concreteness that each X_i is an interval in $\mathbf{R}$. Given is a complete weak order $\succeq$ on X.

Definition 9. *A utility function on X which represents $\succeq$ is* additively separable *if there are functions $u_i : X_i \to \mathbf{R}$ such that*

$$u(x) = u_1(x_1) + \cdots + u_n(x_n)$$

Why does additive separability make sense?

Additive separable representations are "more nearly unique" than ordinal representations. If $U : X \to \mathbf{R}$ is an additive separable representation of $\succ$ and $f : \mathbf{R} \to \mathbf{R}$ is strictly increasing, then $f \circ U$ is a utility representation of $\succ$, but it is not necessarily additively separable.

Theorem 8. *Suppose $U : X \to \mathbf{R}$ is an additively separable representation of $\succ$. The function $V : X \to \mathbf{R}$ is an additively separable representation for $\succ$ iff there are real numbers $a > 0$ and b such that $V = aU + b$.*

This theorem is not true for arbitrary X and U. It requires that the image of X under U be rich enough. See Basu (1982) on this point. The conditions on X we suppose and the conclusions about U we derive will be sufficient to reach this conclusion.

Suppose we can write $X = \prod_{i=1}^{n} X_i$, where each X_i is a connected subset of some Euclidean space. Suppose that $\succ$ is a preference order for which, for all $x \in X$, both $W(x)$ and $B(x)$ are open.

For any subset I of indices and any element $x \in X$, write $x_I = (x_i)_{i \in I}$. Write x_{-i} when referring to the set of all indices but i. Define the preference order $\succ_{x_{I^c}}$ on $\prod_{i \in I} X_i$ such that $a \succ_{x_{I^c}} b$ iff $(a_I, x_{I^c}) \succ (b_I, x_{I^c})$. Think of these orders as preferences on the factors in the list I conditional on receiving the factor bundle x_{I^c}.

Definition 10. *The factors of X are* independent *if for all I and $x, y \in X$, $\succ_{x_I} = \succ_{y_I}$. Factor i is* essential *if there is an x_{-i} such that $\succ_{x_{-i}}$ is non-empty.*

Independence is the requirement that conditional preferences are independent of the factor bundles being conditioned on. Clearly independence is necessary for the existence of an additive representation. If utility is of the form $u(a_I, x_{I^c}) = u_1(a_I) + u_2(x_{I^c})$, then the utility difference $u(a_I, x_{I^c}) - u(b_I, x_{I^c})$ is independent of the factor bundle x_{I^c}.

Theorem 9. *Suppose $\succ$ is a preference order such that the n factors are independent and there are at least three essential factors, then $\succ$ has an additive representation. Each u_i is continuous. The representation is unique up to positive affine transformations.*

Proof. See Debreu (1960) □

This approach to additive separability hides the algebraic structure of the problem in topological assumptions. What guarantees, for instance, the existence of an additive separable representation on a finite set of alternatives?

Here is the "standard" approach, laid out for two factors. Suppose $X = X_1 \times X_2$, and that $\succ$ is a binary relation on X which satisfies the following conditions:

A.1. (preference order): $\succ$ is asymmetric and negatively transitive.

A.2. (independence): For all a, b in X_1 and p, q in X_2, if $ap \succ bp$ then $aq \succ bq$, and if $ap \succ aq$ then $bp \succ bq$.

A.3. (Thomsen): For all $a, b, c \in X_1$ and $p, q, r \in X_2$, if $bp \sim aq$ and $cp \sim ar$, then $cq \sim br$.

A.4. (essential): Both factors are essential.

A.5. (solvability): For $a, b, c \in X_1$ and $p, q, r \in X_2$, if $ap \succeq bq \succeq cp$, then there is an $x \in X_1$ such that $xp \sim bp$, and if $ap \succeq bq \succeq ar$, there is a $y \in X_2$ such that $ay \sim bq$.

A.6. (Archimedes): An Archimedean axiom.

Definition 11. *A pair $(X, \succ)$ is an* additive preference structure *if $X = X_1 \times X_2$ and $\succ$ satisfies axioms A.1–6.*

Theorem 10. *If X is an additive preference structure, then $\succ$ has an additively separable representation, and that representation is unique (among additively separable representations) up to positive affine transformations. If $\succ$ on $X = X_1 \times X_2$ has an additively separable representation, then $\succ$ satisfies A.1–3.*

Proof. A clean proof can be found in Holman (1971). □

Axiom 3 is known as the *Thomsen condition.* The Thomsen condition captures the essence of additive separability. It is easy to check its necessity. It describes a kind of "parallel property" that indifference curves must have. The condition can be described in the figure below.

This figure contains three pairs of points, identifiable by their shading. The Thomsen condition says that if the two points are indifferent in any two of the pairs, the two points in the third pair are indifferent as well. If an indifference curve runs through the two black points, and another runs through both grey points, then a third curve runs through through the two white points. Other condition's can replace the Thomsen condition in theorem 10.

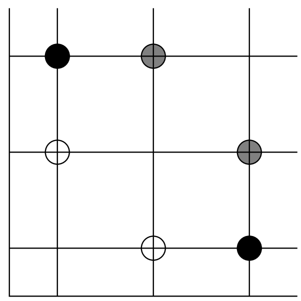

Figure 4: The Thomsen condition.

You might, for instance, Google on the *hexagon condition*.

Exercise 7. *Verify by direct calculation that if* $\succ$ *on* $X_1 \times X_2$ *has an additive separable representation, then the Thomsen condition is satisfied.*

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The Thomsen condition is a statement about how different indifference curves fit together. To see the implications of additive separability for how indifference curves should fit together somewhat differently, take X to be the non-negative orthant of the Euclidean plane, and suppose $\succ$ has a utility representation $U(x, y) = f(x) + g(y)$, and all functions are C^1. The indifference curve corresponding to utility level u is the set of solutions to the equation

$$f(x) + g(y) = u$$

Differentiating implicity, the derivative of the indifference curve in the xy-plane through the point (x, y) is $y'(x) = -f'(x)/g'(y)$. Consider the points A and B in the figure below. The ratio of the slope of the curve through A to that of the curve through B is $g'(y_1)/g'(y_2)$. This expression is independent of x. The points C and D have the same y coordinates as A and B, respectively. So the ratio of the slope of the curve through C to that of the curve through D should be identical. A similar condition must hold for points A and C, and B and D.

Exercise 8. *Take* $X = \mathbf{R}^2_+$, *and define* $U(x,y) = x^2 + xy + y^2$. *The function* U *represents some preference order, and* U *is not additively separable. Does the preference order* U *represents have an additively separable representation? Answer the same question for* $V(x,y) = x^2 + 2xy + y^2$. *Finally, consider* $U_\alpha(x,y) = x^2 + \alpha xy + y^2$ *for* $\alpha \geq 0$. *For which values of the parameter* α *does the preference order represented by* U_α *have an additively separable representation?*

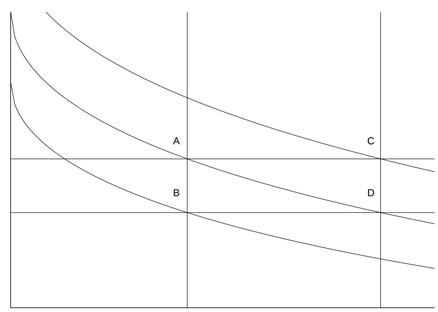

Figure 5: Additive separability slope conditions.

I began this section by claiming that additive separability sits apart from choice under uncertainty. Strictly speaking, this is false. Although additive separability is interesting in many situations where uncertainty plays no role, it has connections to choice under uncertainty as well. A simple example follows. Let's consider bets on whether or not W. will be reelected. A bet can be described by a pair of numbers: what you win if he is reelected, and what you will win if he is not. So for instance, the bet $(10, -10)$ pays off \$10 if W. wins and $-$\$10 if he does not (that is, you pay \$10 if he loses). The bet $(0,0)$ is "no bet". The set of all possible bets is $\mathbf{R}^2$, and a typical bet is the pair (z_1, z_2). An expected utility representation for a preference order $\succ$ on the set of all bets is a pair (p, u) where p is a probability of W. winning, and $u : \mathbf{R} \to \mathbf{R}$ is a real valued function, such that

$$(x_1, x_2) \succ (y_1, y_2) \qquad \text{iff} \qquad pu(z_1) + (1-p)u(z_2) > pu(y_1) + (1-p)u(y_2)$$

That is, p and u are such that the function $U(z_1, z_2) = pu(z_1) + (1-p)u(z_2)$ is a utility representation for $\succ$. Notice that the utility function U is additively separable in its components z_1, z_2. In this case, expected utility is a special case of additive separability on an appropriate set X of choices.

Exercise 9. *Consider the utility function $U(z_1, z_2) = \min(z_1, z_2)$, which in the uncertainty context gives rise to the* maximin criterion. *Which of the assumptions in Theorem 10 does its ordering violate?*

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