Choice Under Uncertainty

- Z a finite set of outcomes.
- P the set of probabilities on Z.
- $p \in P$ is $(p_1, \ldots, p_n)$ with each $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$
- Binary relation $\succ$ on P.

- Objective probability case.
- Decision maker does not care how $p \in P$ is constructed.
- For $\alpha \in [0, 1]$ and $p, q \in P$, $p' \in P$, where

$$p'(z) = \alpha p(z) + (1 - \alpha)q(z)$$

for $z \in Z$.

Expected Utility

An **expected utility representation** of $\succ$ is a $u : Z \to \mathbf{R}$ such that for $p, q \in P$, $p \succ q$ if and only if

$$\sum_{z \in Z} p(z)u(z) > \sum_{z \in Z} q(z)u(z).$$

Example

- $Z = \{\textbf{D}\text{iet coke}, \$1, \textbf{C}\text{oke}\}$.
- Prefer D for sure to \$1 for sure to C for sure, i.e. $(1, 0, 0) \succ (0, 1, 0) \succ (0, 0, 1)$.
- Consider $(0, 1, 0)$ versus $(p_1, 0, 1 - p_1)$.
- Suppose there is a p^* such that $(0, 1, 0) \sim (p^*, 0, 1 - p^*)$.

If there is an EU representation of $\succ$ on P then $u(\$1) = p^* u(D) + (1 - p^*) u(C)$.

Normalize so that $u(D) = 1$ and $u(C) = 0$. Then $u(\$1) = p^*$.

Contrast to a representation of $\succ$ on Z with $D \succ \$1 \succ C$. Any function V such that $V(D) > V(\$1) > V(C)$ will work.

Can set $V(D) = 1$ and $V(C) = 0$, but $V(\$1)$ is any number strictly between 0 and 1.

Axioms

Axiom 1. $\succ$ is a preference relation.

We know that if we have an Archimedean assumption then an ordinal representation of $\succ$ exists. This is a function $V : P \to \mathbf{R}$ such that $p \succ q$ if and only if $V(p) > V(q)$.

We want a particular form for V. There is hope as P is special, not just a set of outcomes, but probabilities on an underlying set of outcomes.

Structure

What does $V(p) = \sum_z p(z)u(z)$ imply about $\succ$?

- Suppose $Z = \{z_1, z_2, z_3\}$. Then any $p \in P$ can be characterized by $\{(p_1, p_3) \in \mathbf{R}^2_+ : p_1 + p_3 \leq 1\}$.
- If we have an EU representation with u, lets write $u(z_i) = u_i$. So $u = (u_1, u_2, u_3)$ is a vector in $\mathbf{R}^3$.
- An indifference curve solves, for a constant c,

  $$c = p_1u_1 + p_2u_2 + p_3u_3 = u_2 - (u_2 - u_1)p_1 + (u_3 - u_2)p_3$$

- So in (p_1, p_3) space indifference curves are parallel lines with slope $(u_2 - u_1)/(u_3 - u_2)$.

Archimedean Axiom

Axiom 2. For all $p, q, r \in P$, if $p \succ q \succ r$ then there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha p + (1-\alpha) r \succ q \succ \beta p + (1-\beta) r.$$

How might this fail?

Suppose r is probability one on an outcome that is so bad that any mix containing it is worse than any mix not containing it.

Independence Axiom

Axiom 3. For $p, q, r \in P$ and $\alpha \in (0, 1]$, if $p \succ q$ then $\alpha p + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$.

Example:

- $Z = \{z_1, z_2, z_3\}$, $p = (1, 0, 0), q = (0, 0, 1)$, $r = (0, 1, 0)$
- $\alpha p + (1 - \alpha) r = (\alpha, 1 - \alpha, 0)$.
- $\alpha q + (1 - \alpha) r = (0, 1 - \alpha, \alpha)$.
- The decision maker will actually receive only one of the outcomes.
- In the α event he prefers the p mixture to the q mixture.
- In the $1 - \alpha$ event he is indifferent as will get r in either mixture.

Is this axiom consistent with observed choice?

Shape of Indifference Curves

Lemma 5.6.c. If $\succ$ on P satisfies Axioms 1, 2 and 3 then, for any $r \in P$,

$$p \sim q \text{ and } \alpha \in [0,1] \implies$$
$$\alpha p + (1-\alpha) r \sim \alpha q + (1-\alpha) r.$$

Von Neumann Morgenstern Theorem

The binary relation $\succ$ on P has an **expected utility representation** if there is a function $u : Z \to \mathbf{R}$ such that for any $p, q \in P$,

$$p \succ q \iff \sum_z u(z)p(z) > \sum_z u(z)q(z).$$

Theorem. A binary relation $\succ$ on P satisfies Axioms 1, 2 and 3 if and only if it has an expected utility representation. Further, if u represents $\succ$ then $u' : Z \to \mathbf{R}$ also represents $\succ$ if and only if there exist numbers $a > 0$ and b such that $u' = au + b$.

Outline of the Proof of the Von Neumann Morgenstern Theorem

1. There are best and worst elements b and w of P. Can focus on the case of $b \succ w$.
2. For any $\alpha, \beta \in [0,1]$, $\beta b + (1-\beta)w \succ \alpha b + (1-\alpha)w$ if and only if $\beta > \alpha$.
3. For any $p \in P$ there is an $\alpha_p \in [0,1]$ such that $\alpha_p b + (1-\alpha_p)w \sim p$.
4. (2) implies that the α_p in (3) is unique.
5. $p \succ q$ if and only if $\alpha_p > \alpha_q$.
6. Let $V(p) = \alpha_p$. By (5) this $V(\cdot)$ represents $\succ$.

7. This $V(\cdot)$ is an affine function, i.e. for any $p, q \in P$ and $\beta \in [0, 1]$ we have $V(\beta p + (1 - \beta)q) = \beta V(p) + (1 - \beta)V(q)$.

8. Now note that any $p \in P$ can be written as a linear combination of sure probabilities on Z.

Let δ_z be a probability that puts unit mass on z.

Then by (7) applied repeatedly, we have $V(p) = \sum_z p(z)V(\delta_z) = \sum_z p(z)u(z)$, where we have defined $u(z) = V(\delta_z)$.

Non-Finite Set of Outcomes

- Let Z be a set of outcomes (not necessarily finite).
- Let P_s be the set of simple probabilities on Z, i.e. those with finite support, $p \in P_s \implies p(z) > 0$ for only a finite number of $z \in Z$.
- The axioms for P_s are unchanged.
- Change P to P_s in the statement of the vNM Theorem.
- Extension to general probabilities is possible.

Risk Aversion

- Outcomes $Z = \mathbf{R}^1$ interpreted as money.
- Let p be a probability on Z, let E_p be the expected value of p and let δ_{E_p} be point mass on E_p.
- Suppose that for all $p \in P$, $\delta_{E_p} \succeq p$. This holds if and only if the utility function u in the vNM theorem is concave.
- The degree of concavity reflects how much the decision maker dislikes risk.
- Cannot measure this with u'' as if u represents $\succ$ so does $v = au + b$ for any $a > 0$.
- Coefficient of Absolute Risk Aversion

$$\lambda(z) = \frac{-u''(z)}{u'(z)}$$

Portfolio Choice

- One risk free asset (money), m, with a total return of 1.
- One risky asset (stock), x, with a Normally distributed total return with mean $\bar{r}$ and variance σ_r^2.
- vNM utility of wealth is $u(z) = -\exp(-\lambda z)$.
- Constant absolute risk aversion $\lambda > 0$.
- If wealth z is normally distributed with mean $\bar{z}$ and variance σ_z^2) then expected utility is

$$-\exp[-\lambda(\bar{z} - \lambda\sigma_z^2/2)].$$

- Let z_0 be initial wealth and p be the price of the risky asset.
- Budget constraint is $z_0 = m + px$.
- Wealth is $z = m + rx = z_0 + x(r - p)$.
- So wealth is Normally distributed with mean $z_0 + x(\bar{r} - p)$ and variance $x^2\sigma_r^2$.

Decision Problem

$$\max_x \; -\exp[-\lambda(\bar{z} - \lambda\sigma_z^2/2)]$$

$\iff$

$$\max_x \; \bar{z} - \lambda\sigma_z^2/2$$

$\iff$

$$\max_x \; z_0 + x(\bar{r} - p) - \lambda x^2\sigma_r^2/2$$

The objective function is concave so the first order condition is necessary and sufficient for a maximum.

$$\bar{r} - p - \lambda x\sigma_r^2 = 0$$

The optimal choice of the amount of risky asset x is

$$x^* = \frac{\bar{r} - p}{\lambda\sigma_r^2}$$