Decision Theory I
Problem Set 4

1. Recall that $f \succ_A g$ iff there is an act h such that $f_A h \succ g_A h$. Show that $\succ_A$ is negatively transitive and asymmetric.

2. Show that if A and B are null, then so is $A \cup B$. (Hint: Prove this first for *disjoint* null events, then extend to all null events. You may use any facts *proved* in the lecture notes.)

3. GRAD: The notes prove the sure thing principle for partitions of size 2. Complete the proof by proving it for partitions of arbitrary size n.

4. GRAD: Assume that the outcomes space is finite. Prove that there are best and worst acts, and that these acts are outcomes. (Recall that we can identify an outcome o with the constant act that gives o in all states.)

5. Suppose a decision-maker is known to have utility linear in money. Suppose that a ball is to be drawn from an urn containing only red, white and blue balls. Suppose that she has a choice of bets on the outcome. Consider the following bets from which she may choose: $b_1 = (1, 4, 4)$, $b_2 = (1, 6, 0)$, $b_3 = (2, 4, 3)$, $b_4 = (3, 3, 4)$ and $b_5 = (4, 5, -1)$, where the bet (x, y, z) pays off x on red, y on white, and z on blue. Suppose she prefers each of b_2, b_3 and b_4 to b_1.

   (a) Are her preferences consistent with expected utility maximization? If so, find a possible probability distribution on balls which could be her beliefs.

   (b) GRAD: Describe all such probability distributions.

   (c) Suppose that, in addition to her preferences above, she also prefers b_1 to b_5. Are her preferences consistent with expected utility maximization? If so, find a possible probability distribution on balls which could be her beliefs.

6. There is a deck with three cards:

   - one is black on both sides,

- one is white on both sides, and
- one is black on one side and white on the other.

Alice chooses a card from the deck and puts it on the table with a black side showing.

(a) What is the probability, according to Bob, that the other side is black? Give at least two answers to the problem, and describe the protocol that generates them.

(b) What if Bob doesn't know Alice's protocol (which is probably the case in practice). What would be a good way to model the problem in that case?

7. You're trying to decide whether or not to spend the morning studying for an afternoon test. You don't particularly like studying, but you definitely want to do well on the test. Suppose for simplicity that you get utility 10 if you don't study and do well, 9 if you study and do well, 0 if you don't study and don't do well, and -1 if you study and don't do well. You have previous experience showing that studying is highly correlated with doing well on tests: the probability of doing well given that you study is .9, and the probability that you do well if you don't study is .1. On the other hand, you didn't get much sleep last night, and you know that typically when you don't sleep well, you neither study (you're too tired) nor do you do well on tests.

   (a) Describe two causal scenarios: one in which not studying causes poor performance on tests and one in which lack of sleep causes both not studying and poor performance. In both scenarios, define causal probabilities that result in the correlation between studying and doing well given above.

   (b) Given the probabilities used in part (a), what is the expected utility of studying in each causal model.

   (c) What information would you need to distinguish the two models?

8. Consider the following Bayesian network containing 3 Boolean random variables (that is, the random variables have two truth values—*true* and *false*):

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   A
  / \
 B   C
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Suppose the Bayesian network has the following conditional probability tables (where X and $\overline{X}$ are abbreviations for $X = \mathit{true}$ and $X = \mathit{false}$, respectively):

$$\Pr(A) = .1$$
$$\Pr(B \mid A) = .7$$
$$\Pr(B \mid \overline{A}) = .2)$$
$$\Pr(C \mid A) = .4)$$
$$\Pr(C \mid \overline{A}) = .6)$$

(a) What is $\Pr(\overline{B} \cap C \mid A)$?

(b) What is $\Pr(A \mid \overline{B} \cap C)$?

(c) Suppose we add a fourth node labeled D to the network, with edges from both B and C to D. For the new network

  (i) Is A conditionally independent of D given B?

  (ii) Is B conditionally independent of C given A?

  In both cases, explain your answer.