Decision Theory I
Problem Set 3

For all of the problems on this problem set you can assume that the axioms we used in proving the Von Neumann Morgenstern Theorem (Theorem 5.4 in Kreps) are satisfied. Note that by Lemma 5.7 of Kreps these axioms imply that there are best and worst elements of the set of prizes X. You can use this lemma without proof.

1. Suppose that the set of prizes is $X = \{1, 4, 9\}$. Suppose that $p = (1/2, 0, 1/2)$, $q = (1/3, 1/3, 1/3)$ and $r = (1/5, 2/5, 2/5)$ are probabilities on X. The individual's preferences $\succ$ have an expected utility representation with utility function over prizes, $u(x) = x^{1/2}$. How does the individual rank p, q and r?

2. Consider a finite set of prizes X and probabilities P on them. Suppose that an individual's preferences $\succ$ on P have an expected utility representation with utility function on prizes $u : X \to \Re$. Show that $\succ$ satisfies the independence axiom.

3. Consider a finite set of prizes X and probabilities P on them. Suppose that an expected utility maximizer's preferences $\succ$ on P have an expected utility representation with utility function on prizes $u : X \to \Re$. Suppose that $v(\cdot) = au(\cdot) + b$ for real numbers $a > 0$ and b. Show that v also represents $\succ$.

4. GRAD Consider a finite set of prizes X and probabilities P on them. Suppose that an expected utility maximizer's preferences $\succ$ on P have an expected utility representation with utility function on prizes $u : X \to \Re$ and an expected utility representation with another utility function on prizes $v : X \to \Re$. Show that there exist real numbers $a > 0$ and b such that $v(\cdot) = au(\cdot) + b$.

5. An expected utility maximizing individual with wealth w will lose $a < w$ with probability p, and will have no loss with probability $1 - p$. Assume that the individual's utility function u satisfies $u''(m) < 0 < u'(m)$ for all $m > 0$. She is offered insurance at a premium of r per unit.

One unit of insurance pays 1 in the event of a loss and 0 otherwise. If she buys x units of insurance her wealth will be $w - a - rx + x$ if there is a loss and $w - rx$ if there is no loss. The insurance is"actuarialy fair", so $r = p$.

(a) Write the individual's expected utility given that x units of insurance are purchased.

(b) How many units of insurance will she purchase? Explain. [Hint: The optimal amount of insurance maximizes the expected utility you wrote in part (a) over the choice of x.]

6. Evaluate the following argument: Max, who is a risk averse expected utility maximizer, is offered an opportunity to buy fair insurance. (The premium equals the expected payoff as in the problem above.) He decides not to buy the insurance. His reasoning is that buying insurance is really gambling; you pay a premium and you may not get a payoff. Fair insurance is a fair gamble, but because he is risk averse he would not accept a fair gamble and thus should not accept fair insurance. [You do not need to provide a proof. It is sufficient to explain whether Max's reasoning is correct or not and why.]

7. Suppose that an expected utility maximizing individual's utility function, $u(\cdot)$, for wealth satisfies $u'(m) > 0 > u''(m)$ for all $m > 0$. The individual's initial wealth is $w > 0$. She is offered a bet with probability 1/2 of winning $2t$ and probability 1/2 of losing t. Assume that $0 < t < w$. Show that if t is small enough then she will accept the bet. [Hint: How does the expected utility change as t changes from $t = 0$ to $t > 0$?]

8. GRAD An expected utility maximizing investor must decide how to allocate his initial wealth $w > 0$ between money and a risky asset. The gross rate of return on money, m, is 1. The gross rate of return on investment in the risky asset, x, is $1 + H > 1$ with probability p and $1 + L < 1$ with probability $1 - p$. So if the investor's portfolio is described by (m, x) his random wealth is $w + Hx$ with probability p and $w + Lx$ with probability $1 - p$. The individual's utility function is u with $u'(w) > 0 > u''(w)$ for all $w > 0$. His choice of x is constrained to lie in $[0, w]$.

(a) Suppose that the expected net rate of return on the asset, $pH + (1-p)L$, is strictly positive. Prove that the individual will hold a strictly positive amount of the risky asset.

(b) Now suppose that $u(w) = \log(w)$. Assume that the optimal amount of the risky asset satisfies $w > x* > 0$. Show that if H increases then so does the optimal amount of the risky asset $x*$. [This, of course, is exactly what you would expect. If in some state you get a higher rate of return on the risky asset while keeping the same probability distribution on returns, you should invest more in the risky asset!]