Decision Theory I
Problem Set 1

Handed out: Sept. 11, 2008. Due: Sept. 25, 2008

1. Show that if $\succ$ is negatively transitive and asymmetric then $\succ$ is transitive.

2. Suppose $X = \{x, y, z\}$. Consider a choice function $C : P(X) \to P(X)$ such that $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y\}$. Does this choice function satisfy Sen's α and β?

3. The set of alternatives is $X = \{a, b, c\}$ and $\succ$ is a binary order on X reflecting strict preference. Suppose that for $x \in \{b, c\}$, $x \not\succ a$ and $a \not\succ x$. Suppose also that $b \succ c$. Can this relation be a strict preference relation? Explain.

   If we want to include the possibility that there is an alternative a that is not comparable to either b or c in our analysis then we would want the condition above on a to be satisfied. What does this example say about non-comparability?

4. Let $\succ$ be a binary relation on a finite set X. Define $\succeq$ by: $x \succeq y$ if $y \not\succ x$. Show

   (a) If $\succeq$ is complete then $\succ$ is asymmetric.

   (b) If $\succeq$ is transitive then $\succ$ is negatively transitive.

5. Suppose that $\succ$ is a partial order and define $c(\cdot, \succ)$ as in class. Show that Sen's axiom α holds, but show by example that Sen's β may fail to hold.

6. **GRAD:** A binary relation that is reflexive, symmetric and transitive is called an equivalence relation. An equivalence relation partitions a set into equivalence classes. Suppose that $\succ$ is a strict preference relation on a finite set X. Then by Proposition 2.4 of Kreps we know that $\sim$ is an equivalence relation on X. For each $x \in X$ define its equivalence class by $I(x) = \{y \in X | y \sim x\}$. Show:

(a) The sets $I(x)$ partition X. (A collection of sets $\{A_1, \ldots, A_N\}$ partitions X if each $x \in X$ is in at least one A_i and $A_i \cap A_j = \phi$ for all $i \neq j$.)

(b) The sets $I(x)$ are strictly ranked. (The equivalence classes are strictly ranked if, for all $x, y \in X$: (1) if $I(x) \neq I(y)$, then either $x \succ y$ or $y \succ x$, and (2) if $x \succ y$ then $x' \succ y'$ for all $x' \in I(x)$ and $y' \in I(y)$.)

7. **GRAD:** In the statement of Sen's α and β we allow the sets A and B to be any subsets of X. So when we proved that these axioms imply that the revealed preference relation is asymmetric and negatively transitive we allowed ourselves to use information about choices from arbitrary subsets of X. We want to know whether there is a smaller class of subsets of X such that the claim in the revealed preference theorem is true if α and β are satisfied on this smaller class of sets. Suppose that the cardinality of X is N and for each integer $n \leq N$ let S_n be the collection of all non-empty subsets of X of cardinality less than or equal to n. Find the smallest $n > 1$ such that the following claim is true: If a choice function satisfies Sen's α and β on S_n then there is a preference order $\succ$ defined on X such that $c(A, \succ) = c(A)$ for all $A \in S_n$.

8. **GRAD:** In class in the proof of the revealed preference theorem we defined strict revealed preference. Weak revealed preference is defined as follows: $x \succeq y$ if $x \in C(\{x, y\})$. Define induced strict revealed preference $\succ^*$ from revealed preference $\succeq$ by: $x \succ^* y$ if $x \succeq y$ and $y \not\succeq x$. Are strict revealed preference and induced strict revealed preference the same relation?